On Packing Colorings of Distance Graphs

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Abstract

The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the least integer k for which there exists a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices of color i are at distance at least $i+1$. This paper studies the packing chromatic number of infinite distance graphs $G(\mathbb{Z}, D)$, i.e. graphs with the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i-j| \in D$. We present lower and upper bounds for $\chi_\rho(G(\mathbb{Z}, D))$, showing that for finite D, the packing chromatic number is finite. Our main result concerns distance graphs with $D = \{1, t\}$ for which we prove some upper bounds on their packing chromatic numbers, the smaller ones being for $t \geq 447$: $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 40$ if t is odd and $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 81$ if t is even.

Keywords: graph coloring; packing chromatic number; distance graph.

1 Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A *packing k-coloring* of a graph G is a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices of color i are at distance at least $i+1$ (thus vertices of color i form an i-packing of G). The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k for which G has a packing k-coloring.

This parameter was introduced recently by Goddard et al. [9] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. Fiala and Golovach [6] showed that the problem remains NP-complete for trees. Brešar et al. [2] studied the problem on Cartesian products graphs, hexagonal lattice and trees, using the name of packing chromatic number. Other studies on this parameter mainly concern infinite graphs, with a natural question to be answered : *is a given infinite graph has finite packing chromatic number ?* Goddard et al. answered by the positive for the infinite two dimensional square grid by showing $9 \leq \chi_\rho \leq 23$. The lower bound was later improved to 10 by Fiala et al. [7] and then to 12 by Ekstein et al. [5]. The upper bound was recently improved by Holub and Soukal [13] to 17. Fiala et al. [7] showed that the infinite hexagonal grid has packing chromatic number 7; while the infinite triangular lattice along with the 3-dimensional square lattice was shown to admit no finite packing coloring by Finbow and Rall [8]. Infinite product graphs were considered by Fiala et al. [7] that showed that the product of a finite path (of

order at least two) by the 2-dimensional square grid has infinite packing chromatic number while the product of the infinite path by any finite graph has finite packing chromatic number.

The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \ldots, d_k\}$, where d_i being positive integers, has the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| \in D$. The *finite distance graph* $G_n(D)$ is the subgraph of $G(\mathbb{Z}, D)$ induced by vertices $0, 1, \ldots, n-1$. An edge between vertices a and $a + d_i$ will be called a d_i-edge.

The study of distance graphs was initiated by Eggleton et al. [3]. A large amount of work has focused on colorings of distance graphs [4, 16, 1, 11, 12, 14], but other parameters have also been studied on distance graphs, like the feedback vertex set problem [10].

The aim of this paper is to study the packing chromatic number of infinite distance graphs, with particular emphasis on the case $D = \{1, t\}$. In section 2, we bound the packing chromatic number of the infinite path power (i.e. infinite distance graph with $D = \{1, 2, \ldots, t\}$). Section 3 concerns packing colorings of distance graphs with $D = \{1, t\}$, for which we prove some lower and upper bounds on the number of colors (see Proposition 1). Exact or sharp results for the packing chromatic number of some other 4-regular distance graphs are presented in Section 4. Section 5 concludes the paper with some remarks and open questions.

Our results about the packing chromatic number of $G(\mathbb{Z}, D)$ for some small values of D (from Sections 2 and 4) are summarized in Table 1. The source code (`C++`) of the program used to obtain most of the lower bounds and some upper bounds along with long sequences of colors not given in this paper can be found at [15].

D	$\chi_\rho \geq$	$\chi_\rho \leq$	period
$1, 2$	8*	8	54
$1, 3$	9*	9	32
$1, 4$	11	16	320
$1, 5$	10*	12	1028
$1, 6$	11*	23	1917
$1, 7$	10*	15	640
$1, 8$	11*	25	5184
$1, 9$	10*	18	576
$1, 2, 3$	19	23	768
$2, 3$	11	13	240
$2, 4$	8*	8	54
$2, 5$	14	23	336
$2, 6$	9*	9	32

Table 1: Lower and upper bounds for the packing chromatic number of $G(\mathbb{Z}, D)$ for different values of D. In the fourth column are the periods of the colorings giving the upper bounds. (*: bound obtained by a computer search).

The bounds of Section 3 are summarized in the following Proposition:

Proposition 1.

$$\chi_\rho(G(\mathbb{Z},\{1,t\})) \leq \begin{cases} 86, & t = 2q+1, q \geq 36 \\ 40, & t = 2q+1, q \geq 223 \\ 173, & t = 2q, q \geq 87 \\ 81, & t = 2q, q \geq 224 \\ 29, & t = 96q \pm 1, q \geq 1 \\ 59, & t = 96q + 1 \pm 1, q \geq 1 \end{cases}$$

2 Path Powers

Let $D^t = G(\mathbb{Z},\{1,2,\ldots,t\})$ be the t^{th} power of the two-ways infinite path and let $P_n^t = G_n(\{1,2,\ldots,t\})$ be the t^{th} power of the path P_n on n vertices.

We first present an asymptotic result on the packing chromatic number:

Proposition 2. *$\chi_\rho(D^t) = (1+o(1))3^t$ and $\chi_\rho(D^t) = \Omega(e^t)$.*

Proof. D^t is a spanning subgraph of the lexicographic product $\mathbb{Z} \circ K_t$ (see Figure 1). Then, as Goddard et al. [9] showed that $\chi_\rho(\mathbb{Z} \circ K_t) = (1+o(1))3^t$, the same upper bound holds for D^t. To prove the lower bound, since two vertices of color i have to be $it+1$ apart, then for any packing coloring of D^t using at most c colors, c must satisfy:

$$\sum_{i=1}^{c} \frac{1}{it+1} \geq 1.$$

Since

$$\sum_{i=1}^{c} \frac{1}{it+1} < \sum_{i=1}^{c} \frac{1}{it} = \frac{1}{t}\sum_{i=1}^{c} \frac{1}{i} = \frac{H_c}{t},$$

where H_n is the nth harmonic number and since $H_n = \Omega(\ln(n))$, then $\frac{H_c}{t} \geq 1$ implies $c = \Omega(e^t)$.

□

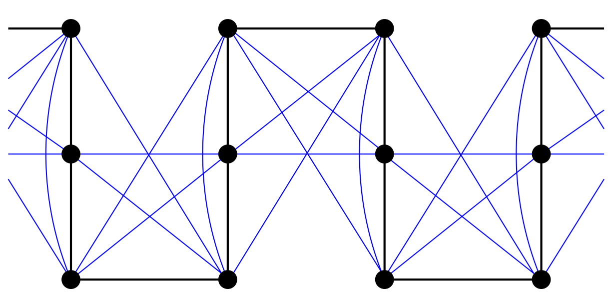

Figure 1: The infinite distance graph D^3 as a subgraph of the lexicographic product $\mathbb{Z} \circ K_3$.

Corollary 1. *For any finite subset D of $\mathbb{N}$, the packing chromatic number of $G(\mathbb{Z}, D)$ is finite.*

For very small t, exact values or sharp bounds for the packing chromatic number can be calculated:

Proposition 3.
$$\chi_\rho(D^2) = 8.$$

Proof. A periodic packing coloring of period 54 using 8 colors is given by the following sequence of length 54 :

$$8,1,2,6,1,4,3,2,1,5,7,1,2,3,4,1,6,2,1,8,3,1,2,4,1,5,7,$$
$$1,3,2,1,6,4,1,2,3,1,8,5,1,2,4,1,3,6,1,2,7,1,5,4,2,1,3.$$

One can check that the distance (considered cyclically) between two occurrences of a color i in this sequence is at least $2i+1$, hence this coloring is a packing coloring of D^2.

Moreover, by checking all cases with the help of a computer, we find that 7 colors are not sufficient for a packing coloring of P^2_{26}. □

Proposition 4.
$$19 \leq \chi_\rho(D^3) \leq 23.$$

Proof. The upper bound comes from a coloring of period 768 using 23 colors described by the sequence of length 768 of Appendix B1.

To prove the lower bound, we consider the maximum density ρ_i of a color i in a packing coloring of D^3. As $d(j,k) = \lceil \frac{k-j}{3} \rceil$, then $\rho_i = \frac{1}{3i+1}$. However, for the colors 1 and 2, we show that only at most 10 over 28 consecutive vertices of D^3 can be colored with these colors: at most $28/4 = 7$ vertices can be colored 1 and at most $28/7 = 4$ vertices can be colored 2, but as $\mathrm{lcm}(4,7) = 28$, we have to choose between color 1 and color 2 for at least one vertex, thus at most $7+4-1 = 10$ vertices can be colored 1 or 2. Then, an easy computation gives that $\chi_\rho(D^3) \geq \min\{c,\ \frac{10}{28} + \sum_{i=3}^{c} \frac{1}{3i+1} \geq 1\} = 19$. □

3 Distance graphs with $D = \{1, t\}$

Let $D(a,b) = G(\mathbb{Z}, \{a,b\})$ be the infinite distance graph with chord lengths a and b. Let also $D_n(a,b) = G_n(\{a,b\})$.

The case $a = 1$ and $b = 2$ was discussed in the previous section, so we now consider $D(1,t)$ with $t \geq 3$.

The general method we shall use will be to cut the graph into blocks B_i of size $s = t-1$ or $s = t+1$, depending on the value of t and to color each block by a predefined color pattern. Figure 2 illustrate the grid-like structure of $D(1,t)$.

Let $s = t \pm 1$ and let $B_i = \{is, is+1, \ldots, (i+1)s-1\}$. Then $D(1,t) = \cup_{i=-\infty}^{+\infty} B_i$. Remark that if $t = 4p-1 = s-1$, then each B_i is an induced cycle of $D(1,t)$ of length $s = t+1 = 4p$ (see Figure 2). By a color pattern P, we mean a sequence of s colors $(c_1, c_2, \ldots, c_s)$ (that will be associated to a block B_i). If S is a sequence of integers, S^p is the sequence obtained by repeating p times S.

We first need to know the distance between two vertices in $D(1,t)$.

Lemma 1. *The distance between two vertices a and b of $D(1,t)$ is $d(a,b) = \min(q+r, q+1+t-r)$, where $|b-a| = qt+r$, with $0 \leq r < t$.*

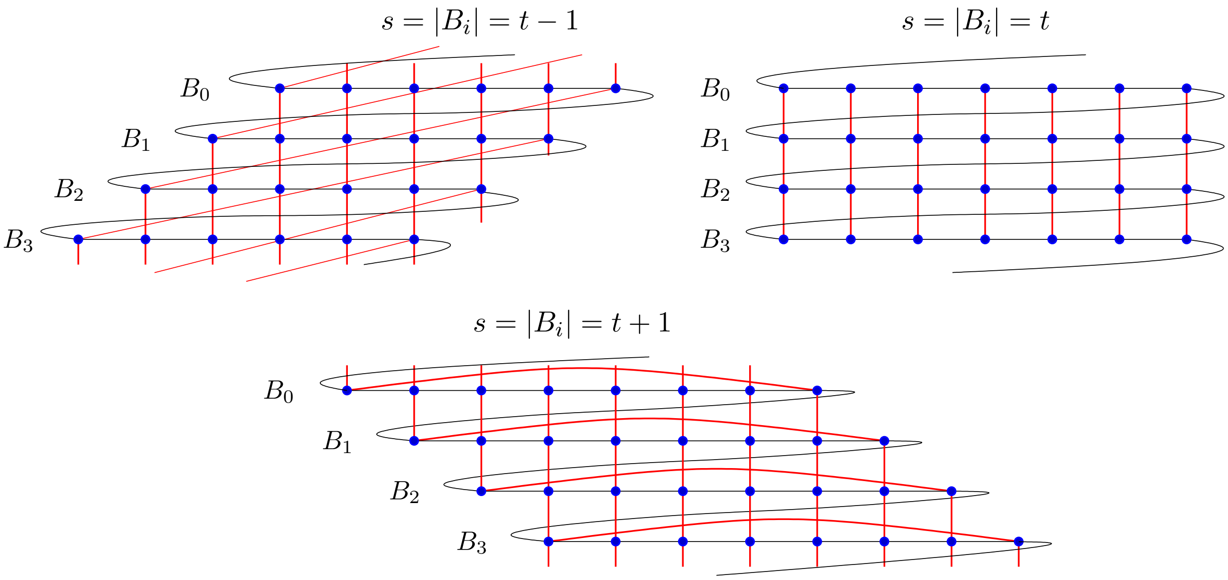

Figure 2: Three block-representations of $D(1, t)$

Proof. Assume w.l.o.g. that $b \geq a$. Any minimal path between a and b uses either q t-edges and r 1-edges or $q + 1$ t-edges and $t - r$ 1-edges. □

The key Lemma of our method is the following which indicates when a color of a pattern can be re-used on another pattern.

Lemma 2. *Let $t = 4p - 1$ or $4p$, $s = t + 1$ and let P, P' be two color patterns containing the color $m < t$ at the same places and no common color greater than m. If P lies on block B_i of $D(1, t)$, then P' can be placed on block B_j whenever $|j - i| > \frac{m}{2}$.*

Proof. Assume without loss of generality that $i = 0$. Let m be a color (an integer) lying on block B_0 (perhaps in several places) and on block B_j (and not on blocks B_h, $0 < h < j$). Let $a_1, a_2, \ldots, a_d$ be the positions of the occurrences of color m in B_0 and let $b_1, b_2, \ldots, b_d$ be the positions of the occurrences of color m in B_j. Then $b_k = a_k + js$.

We are going to show that for any $1 \leq k, \ell \leq d$, we have $d(a_k, b_\ell) > m$ whenever $j > \frac{m}{2}$. For this, we distinguish two cases depending on the values of k and ℓ.

Case 1. $k = \ell$. Then $b_\ell - a_k = js = jt + j$. Hence, by virtue of Lemma 1, $d(a_k, b_\ell) = \min(j + j, j + 1 + t - j) = \min(2j, t + 1) > m$ as soon as $2j > m$, i.e. $j > \frac{m}{2}$.

Case 2. $k \neq \ell$. Let $\delta_b = |b_\ell - b_k|$. Then, as the coloring is a packing coloring of B_j (which is a cycle of length $t + 1$), we have

$$(\ell - k)m < \delta_b < t - m + 1 \tag{1}$$

Subcase 2.1. $k < \ell$. Then $b_\ell - a_k = jt + j + \delta_b$.

If $j + \delta_b < t$, then $b_\ell - a_k = jt + j + \delta_b$. Hence, Lemma 1 gives $d(a_k, b_\ell) = \min(j + j + \delta_b, j + 1 + t - j - \delta_b) = \min(2j + \delta_b, t + 1 - \delta_b) > m$ since $j > \frac{m}{2}$ and $t + 1 - \delta_b > m$ by (1).

If $j + \delta_b \geq t$, then $b_\ell - a_k = (j + 1)t + (j + \delta_b - t)$. Hence, Lemma 1 gives $d(a_k, b_\ell) = \min(j + 1 + j + \delta_b - t, j + 2 + t - j - \delta_b + t) = \min(2j + 1 + \delta_b - t, 2t + 2 - \delta_b) > m$ since $2j + 1 + \delta_b - t = j + 1 + j + \delta_b - t \geq j + 1 \geq t - \delta_b + 1 > m$ and $2t + 2 - \delta_b \geq t + 2 > m + 2$.

Subcase 2.2. $k > \ell$. Then $b_\ell - a_k = js - (b_k - b_\ell) = jt + j - \delta_b$.

If $\delta_b < j$, then $b_\ell - a_k = jt + (j - \delta_b)$. Hence, Lemma 1 gives $d(a_k, b_\ell) = \min(j + j - \delta_b, j + 1 + t - j + \delta_b) = \min(2j - \delta_b, t + 1 + \delta_b) > m$ since $j \geq \delta_b > m$ and $t + 1 + \delta_b > t > m$.

If $\delta_b > j$, then $b_\ell - a_k = (j - 1)t + (t + j - \delta_b)$. Hence, Lemma 1 gives $d(a_k, b_\ell) = \min(j - 1 + t + j - \delta_b, j + t - t - j + \delta_b) = \min(2j + t - 1 - \delta_b, \delta_b) > m$ since by (1), $t - \delta_b > m - 1$ and $\delta_b > m$.

See Figure 3 for an illustration when $t = 7$.

□

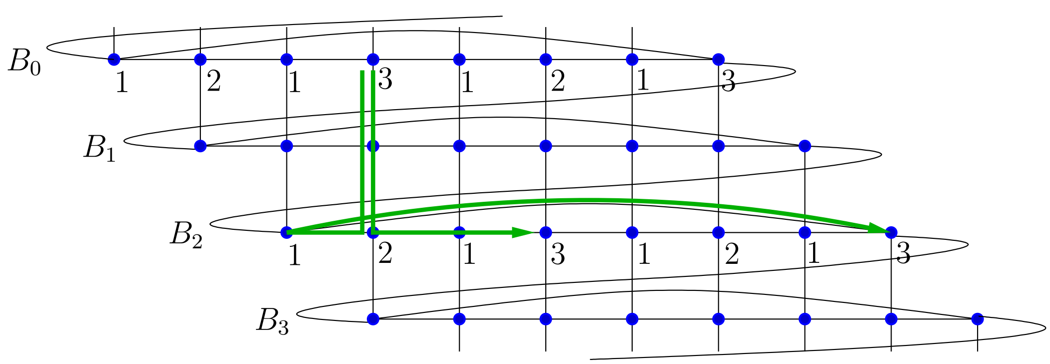

Figure 3: Two shortest paths in $D(1, 7)$ between a vertex colored 3 in block B_0 and two other vertices colored 3 in B_2.

The next result shows that a packing coloring of $D(1, t)$ by patterns that satisfies conditions of Lemma 1 is also a packing coloring of $D(1, t + 2)$.

Lemma 3. *for $t = 4p - 1$ or $4p$, any pattern packing coloring of $D(1, t)$ is a packing coloring of $D(1, t + 2)$.*

Proof. Let $t = 4p - 1$ (resp. $4p$) and f be a pattern packing coloring of $D(1, 4p - 1)$ (resp. $D(1, 4p)$). Assume $D(1, 4p + 1)$ (resp. $D(1, 4p + 2)$) is now colored by f. Fix $t' = t + 2$. Then we have to check the two cases of the proof of Lemma 2 with $m < t = t' - 2$ (and $s = t + 1 = t' - 1$).

Case 1. $k = \ell$. Then $b_\ell - a_k = js = jt' - j = (j - 1)t' + t' - j$. Hence $d(a_k, b_\ell) = \min(j - 1 + t' - j, j + t' - t' + j) = \min(t' - 1, 2j) > m$ since $m < t' - 2$ and $2j > m$.

Case 2. $k \neq \ell$. Let $\delta_b = |b_\ell - b_k|$. Since $t = t' - 2$, then Equation (1) now becomes

$$(\ell - k)m < \delta_b < t' - m - 1 \tag{2}$$

Subcase 2.1. $k < \ell$. Then $b_\ell - a_k = jt' - j + \delta_b$.

If $\delta_b - j \geq 0$, then we have $d(a_k, b_\ell) = \min(j - j + \delta_b, j + 1 + t' + j - \delta_b) = \min(\delta_b, 2j + 1 + t' - \delta_b) > m$ since $\delta_b > m$ and $t' - \delta_b > m + 1$ by (2).

If $\delta_b - j < 0$, then $j > \delta_b > m$. In that case, we have $b_\ell - a_k = (j-1)t' + (t' - j + \delta_b)$ and $d(a_k, b_\ell) = \min(j - 1 + t' - j + \delta_b, j + t' - t' + j - \delta_b) = \min(t' - 1 + \delta_b, 2j - \delta_b) > m$ since by hypothesis, $j > \delta_b > m$.

Subcase 2.2. $k > \ell$. Then $b_\ell - a_k = js - \delta_b = jt' - j - \delta_b$.

If $j + \delta_b \leq t'$, then $b_\ell - a_k = (j-1)t' + (t' - j - \delta_b)$ and $d(a_k, b_\ell) = \min(j - 1 + t' - j - \delta_b, j + t' - t' + j + \delta_b) = \min(t' - 1 - \delta_b, 2j + \delta_b) > m$ since $t' - 1 - \delta_b > m$ by (2).

If $j + \delta_b > t'$, as $\delta_b < t' - m - 1$, then $j > m + 1$. We have $b_\ell - a_k = (j-2)t' + (2t' - j - \delta_b)$ and $d(a_k, b_\ell) = \min(j - 2 + 2t' - j - \delta_b, j - 1 + t' - 2t' + j + \delta_b) = \min(2t' - 2 - \delta_b, 2j - t' - 1 + \delta_b) > m$ since $2j - t' - 1 + \delta_b = j + \delta_b - t' + j - 1$ and $j - 1 > m$ in that case.

See Figure 4 for an illustration when $t = 9$. □

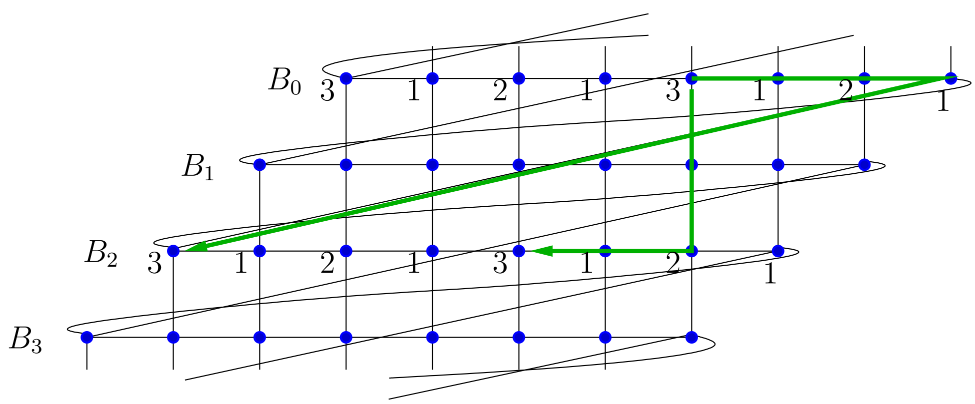

Figure 4: Two shortest paths in $D(1, 9)$ between a vertex colored 3 in block B_0 and two other vertices colored 3 in B_2.

3.1 $D(1, t)$ with odd t

We now present a construction to obtain a packing coloring of $D(1, t)$ for $t \equiv \pm 1 \bmod 96$, with at most 29 colors. To prove this inequality, we shall construct a packing coloring by defining a set of color patterns that will be used cyclically to color the blocks of $D(1, t)$.

Proposition 5. *For $t \equiv -1 \bmod 96$ or $t \equiv 1 \bmod 96$,*

$$\chi_\rho(D(1, t)) \leq 29.$$

Proof. Let $s = 96q$ for some $q \geq 1$ and $t = 96q \pm 1$. We use the following color patterns:

$P_1 = (1, 2, 1, 3)^{24q}$
$P_2 = (1, 4, 1, 5, 1, 8, 1, 4, 1, 5, 1, 9)^{8q}$
$P_2' = (1, 4, 1, 5, 1, 10, 1, 4, 1, 5, 1, 11)^{8q}$
$P_3 = (1, 6, 1, 7, 1, 12, 1, 13, 1, 6, 1, 7, 1, 14, 1, 15)^{6q}$
$P_3' = (1, 6, 1, 7, 1, 16, 1, 17, 1, 6, 1, 7, 1, 18, 1, 19, 1, 6, 1, 7, 1, 20, 1, 21)^{4q}$
$P_3'' = (1, 6, 1, 7, 1, 22, 1, 23, 1, 6, 1, 7, 1, 24, 1, 25, 1, 6, 1, 7, 1, 26, 1, 27, 1, 6, 1, 7, 1, 28, 1, 29)^{3q}$

Then, a packing coloring of $D(1, t)$ using these patterns is constructed by assigning inductively to 16 consecutive blocks B_i the pattern

$$\mathcal{P} = (P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3', P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3'').$$

By virtue of Lemmas 1 and 2, we only have to show that color patterns with common colors are separated enough in $\mathcal{P}$: the distance (considered cyclically) between two patterns P and P' with a maximum common color m in $\mathcal{P}$ has to be greater than $\frac{m}{2}$. It is easily seen that this is effectively the case, as indicated in the following table:

Patterns	Cyclic distance	Max common color
P_1, P_1	2	3
P_2, P_2	8	9
P_2', P_2'	8	11
P_2, P_2'	4	5
P_3, P_3	8	15
P_3', P_3'	16	21
P_3'', P_3''	16	29
P_3, P_3'	4	7
P_3, P_3''	8	7
P_3', P_3''	4	7

Hence, the coloring is a packing coloring of $D(1, t)$ with 29 colors. □

We now generalize this method to color any distance graph $D(1, t)$ with sufficiently large odd t.

Proposition 6. *For any odd $t \geq 73$,*

$$\chi_\rho(D(1, t)) \leq 86.$$

Proof. The method we are going to use to define a packing coloring of $D(1, t)$ with at most 86 colors (the exact number of colors used will vary between 31 and 86, depending on the residue of s modulo 48) is similar with the one of proof of Proposition 5: $D(1, t)$ will be colored by the pattern (of color patterns) $\mathcal{P} = (P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3', P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3'')$ defined in proof of Proposition 5, where the P_i, P_i' and P_i'' will be defined depending on the residue of s modulo 48.

Our base case is $s \equiv 0 \bmod 48$, *i.e.* $t = 48q \pm 1$, for which we use the following color patterns (which are similar with the color patterns for $t = 96q \pm 1$, except for P_3''):

P_1	$(1,2,1,3)^{12q}$
P_2	$(1,4,1,5,1,8,1,4,1,5,1,9)^{4q}$
P_2'	$(1,4,1,5,1,10,1,4,1,5,1,11)^{4q}$
P_3	$(1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^{3q}$
P_3'	$(1,6,1,7,1,16,1,17,1,6,1,7,1,18,1,19,1,6,1,7,1,20,1,21)^{2q}$
P_3''	$(1,6,1,7,1,22,1,23,1,6,1,7,1,24,1,25,1,6,1,7,1,26,1,27,$ $1,6,1,7,1,22,1,23,1,6,1,7,1,28,1,29,1,6,1,7,1,30,1,31)^{q}$

Now, for $s \equiv 4 \bmod 48$ with $q \geq 2$, we are going to modify the above patterns by adding some new colors (from 32 to 58) at the end of the patterns, as indicated in the following table:

P_1	$(1,2,1,3)^{12q},1,2,1,3$
P_2	$(1,4,1,5,1,8,1,4,1,5,1,9)^{4q},1,\left\{\begin{smallmatrix}32\\33\\34\end{smallmatrix}\right.,1,\left\{\begin{smallmatrix}35\\36\\37\end{smallmatrix}\right.$
P_2'	$(1,4,1,5,1,10,1,4,1,5,1,11)^{4q},1,\left\{\begin{smallmatrix}38\\39\\40\end{smallmatrix}\right.,1,\left\{\begin{smallmatrix}41\\42\\43\end{smallmatrix}\right.$
P_3	$(1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^{3q},1,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,1,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$
P_3'	$(1,6,1,7,1,16,1,17,\ldots,1,6,1,7,1,20,1,21)^{2q},1,\left\{\begin{smallmatrix}51\\52\end{smallmatrix}\right.,1,\left\{\begin{smallmatrix}53\\54\end{smallmatrix}\right.$
P_3''	$(1,6,1,7,1,22,1,23,\ldots,1,6,1,7,1,30,1,31)^{q},1,\left\{\begin{smallmatrix}55\\56\end{smallmatrix}\right.,1,\left\{\begin{smallmatrix}57\\58\end{smallmatrix}\right.$

For instance for P_2, two new colors from $\{32,33,\ldots,37\}$ are used in turn, i.e. the last four integers of successive occurrences of P_2 will be $1,32,1,35$; $1,33,1,36$; $1,34,1,37$; $1,32,1,35$; .... As the period of the pattern P_2 in the coloring is 8, then two patterns ending, say, by $1,34,1,37$ will be repeated each 24 patterns and thus Lemma 2 asserts that two occurrences of the color 34 are separated quite enough and the same goes for the color 37.

As can be seen, seven new colors are used for P_3, since if only six new colors were used, the patterns containing the color 48 (or 49) will be repeated each 24 times, but it is not sufficient to ensure that vertices colored by 48 (or 49) are separated quite enough. Then, the last four integers of successive occurrences of P_3 will be $1,42,1,47$; $1,43,1,48$; $1,44,1,49$; $1,42,1,50$; $1,43,1,47$, ... Therefore, two occurrences of P_3 will end by the same four colors each 12 times and will end by the same color (from $\{47,48,49,50\}$) each 4 times (hence each 32 sequences). With Lemma 2, we know that it will not cause any conflict if the colors used are less than $32*2=64$.

For the other residues of s modulo 48 (with $q \geq 2$ if $s \bmod 48 < 24$), the added colors (to the base case $s \equiv 0 \bmod 48$) are given in the four tables of Appendix A (without the 1s, for sake of brevity). Notice also that for the case $s \equiv 24 \bmod 48$, the colors 22 and 23 are reused to complete the pattern P_3''. □

As the next Proposition shows, increasing the minimum value of t allows to shorten the number of colors for a packing coloring of $D(1,t)$.

Proposition 7. *For any odd $t \geq 447$,*

$$\chi_\rho(D(1,t)) \leq 40.$$

Proof. The main idea to obtain a packing coloring of $D(1,t)$ is to modify the coloring of $D(1,t)$ for $s \equiv 0 \bmod 96$ given in Proof of Proposition 5 by adding only one new color α_i to each block B_i. In order to do that, depending on the value of s, α_i must be placed in several

(quasi evenly distributed) positions in block B_i. The conditions for the coloring to remain a packing coloring are i) that vertices of B_i colored α_i have to be (cyclically) distant quite enough and that ii) the color α_i is not reused in another block B_j with $|j-i| \leq \frac{\alpha_i}{2}$ (necessary condition for Lemma 2).

Color patterns are modified in this way:

$P_1 = (1,2,1,3)^{q_1}, s = 4q_1$

$P_2 = (1,4,1,5,1,8,1,4,1,5,1,9)^{q_2} \sqcup (1, \left\{ \begin{smallmatrix} 32 \\ 33 \\ 34 \end{smallmatrix} \right.)^{r_2}, s = 12q_2 + 2r_2, 0 \leq r_2 \leq 4$

$P_2' = (1,4,1,5,1,10,1,4,1,5,1,11)^{q_2} \sqcup (1, \left\{ \begin{smallmatrix} 35 \\ 36 \\ 37 \end{smallmatrix} \right.)^{r_2}, s = 12q_2 + 2r_2, 0 \leq r_2 \leq 4$

$P_3 = (1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^{q_3} \sqcup (1, \left\{ \begin{smallmatrix} 38 \\ 39 \\ 40 \end{smallmatrix} \right.)^{r_3}, s = 16q_3 + 2r_3, 0 \leq r_3 \leq 6$

$P_3' = (1,6,1,7,1,16,1,17,\dots,1,6,1,7,1,20,1,21)^{q_4} \sqcup (1,30)^{r_4}, s = 24q_4 + 2r_4, 0 \leq r_4 \leq 10$

$P_3'' = (1,6,1,7,1,22,1,23,\dots,1,6,1,7,1,28,1,29)^{q_5} \sqcup (1,31)^{r_5}, s = 32q_5 + 2r_5, 0 \leq r_5 \leq 14,$

where $S \sqcup (1,\alpha)^r$ is a sequence obtained by inserting r quasi evenly cyclically distributed occurrences of the pair $(1,\alpha)$ in the sequence S (insertions are possible only after a color > 1, in order to keep the sequence alternate between color 1 and other colors).

For example, $(1,4,1,5,1,8,1,4,1,5,1,9)^3 \sqcup (1,\alpha)^5$ can be rewritten as

$(1,4,1,5,1,8,1,\alpha,1,4,1,5,1,9,1,\alpha,1,4,1,5,1,8,1,4,1,\alpha,1,5,1,9,1,4,1,5,1,\alpha,1,8,1,4,1,5,1,9,1,\alpha)$.

In order to satisfy Condition i) and as the pairs $(1,\alpha)$ have to be inserted only on even positions, we must have $2\lfloor\lfloor\frac{|S|}{r}\rfloor/2\rfloor \geq \alpha$. Hence the worst case for this separation constraint is for the color 31 in P_3'' when $r_5 = 14$: one can insert 14 occurrences of $(1,31)$ if $2\lfloor\lfloor\frac{32q_5}{14}\rfloor/2\rfloor \geq 31$, which is true as soon as $q_5 = 14$ and thus $s = 448$.

Moreover, it can be seen that the added color in each pattern is chosen in such a way that Condition ii) is satisfied. For P_2, colors 32, 33 and 34 will be used in turn (i.e. the first block colored by P_2 will use color 32, the second 33, the third 34 and so on... And the same goes for P_2' and P_3. The patterns P_3' (P_3'', respectively) are distant quite enough in $\mathcal{P}$ to use always the same new color (30 and 31, respectively).

□

Remark that the above method can produce a packing coloring using less than 40 colors, depending on the value of s (i.e. if some r_i are equal to zero). Notice also that combining the methods of Proposition 6 and 7 allows to define a packing coloring for $95 \leq t \leq 447$ using a number of colors lying between 40 and 86.

3.2 $D(1,t)$ with even t

In this subsection, we adapt the method of the previous subsection to obtain upper bounds for the packing chromatic number of $D(1,t)$ when t is even. Although the main idea is the same, it is more complicated (and much more colors are needed) than for the odd case because of the fact that one cannot alternate between color one and other colors too many times (at most $t/2$ times).

The distance graph $D(1,t)$, with $t = 4p$ or $4p+2$ is cut in blocks $B_0, B_1, \dots$ of size $s = 4p+1$ and new color patterns are constructed by inserting a new color at the end of each pattern (of length $s' = s-1$) of Proofs of Propositions 5, 6 and 7.

Proposition 8. *For any even t,*

- *if $t \geq 174$, then $\chi_\rho(D(1,t)) \leq 173$;*

- *if* $t \geq 448$*, then* $\chi_\rho(D(1,t)) \leq 81$;
- *if* $t \equiv 0$ *or* $2 \bmod 96$*, then* $\chi_\rho(D(1,t)) \leq 59$.

Proof. By Lemmas 2 and 3 and by Propositions 5, 6 and 7, it remains to add the missing color in each color pattern. This problem is equivalent to the one of coloring the infinite path P_∞ with colors from $\{k_1, k_1+1, \ldots, k_2\}$ such that vertices of color i are at distance greater than $\frac{i}{2}$ (and with $t > k_2$).

We are going to show, by induction on k_1, that $k_2 \leq 2k_1 - 1$. For $k_1 = 2$, vertices can be colored by alternating color 2 and color 3, so $k_2 = 3$. Assume that P_∞ can be colored with colors from $\{k_1, k_1+1, \ldots, k_2 \leq 2k_1 - 1\}$ and let $k'_1 = k_1 + 1$. Replace now color k_1 by colors k_2+1 and k_2+2 alternatively. Then the maximum color used is $k'_2 = k_2+2 \leq 2k_1+1 = 2k'_1-1$ and the constraint is satisfied since if vertices x and y are colored k_2+2 then their mutual distance satisfies $d(x,y) > 2\frac{k_1}{2} \geq \frac{k_2+1}{2} > \frac{k_2}{2}$.

As the colorings defined in Proof of Proposition 6 (Propositions 7 and 5, respectively) use at most 86 (40 and 29, respectively) colors, then we obtain a packing coloring of $D(1,2t)$ with at most $2 \times 87 - 1 = 173$ colors (81 and 59, respectively).

□

In fact, it seems that less than $2k_1 - 1$ colors are enough for such a coloring. When $k_1 = 87$, a computation gives $k_2 = 152$ for such a coloring; when $k_1 = 41$, we find $k_2 = 71$ and when $k_1 = 30$, we find $k_2 = 53$.

4 $D(a,b)$ with small a and b

Results of Section 3 do not apply for $D(1,t)$ with small t, however it is possible to derive exact or sharp results for such graphs, using ad-hoc methods.

Proposition 9.

$$\chi_\rho(D(1,3)) = 9.$$

Proof. first, remark that the graph-distance $d(i,j)$ between vertex i and vertex $j \geq i$ is $d(i,j) = \lfloor \frac{j-i}{3} \rfloor + (j-i) \bmod 3$.

A 13-packing coloring of $D(1,3)$ of period 32 is given by the following sequence:
$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,4,1,5,1,2,1,3,1,8,1,9$.

It is routine to check that vertices of a same color are distant quite enough.

Again, with the help of a computer, we find that 8 colors are not sufficient for a packing coloring of $D_{100}(1,3)$.

□

Proposition 10.

$$11 \leq \chi_\rho(D(1,4)) \leq 16;$$
$$10 \leq \chi_\rho(D(1,5)) \leq 12;$$
$$11 \leq \chi_\rho(D(1,6)) \leq 23;$$
$$10 \leq \chi_\rho(D(1,7)) \leq 15;$$
$$11 \leq \chi_\rho(D(1,8)) \leq 25;$$
$$10 \leq \chi_\rho(D(1,9)) \leq 18.$$

Proof. For $D(1,4)$, the lower bound is obtained by calculating the maximum density ρ_i of a color i: it can be seen that $\rho_1 = \frac{2}{5}$ and $\rho_i = \frac{1}{4i-2}$ for $i \geq 2$ and that $\min\{c,\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{4i-2} \geq 1\} = 11$. For the upper bound, a 16-packing coloring of period 320 is given in Appendix B2.

For $D(1,5)$, the computer tells us that there exists no 9-packing coloring of $D_{45}(1,5)$ and a 12-packing coloring of period 1028 is given in [15].

Similarly, for $D(1,6)$, there exists no 10-packing coloring of $D_{45}(1,6)$ and a 23-packing coloring of period 1917 is given in [15]. For $D(1,7)$, there exists no 9-packing coloring of $D_{45}(1,7)$ and a 15-packing coloring of period 640 is given in [15]. For $D(1,8)$, there exists no 10-packing coloring of $D_{45}(1,8)$ and a 25-packing coloring of period 5184 is given in [15]. For $D(1,9)$, there exists no 9-packing coloring of $D_{45}(1,7)$ and a 18-packing coloring of period 576 is given in [15].

□

It is interesting to notice that sometimes adding just one more color allows to shorten considerably the period of the packing coloring, as can be seen with $D(1,5)$ with the following periodic 13-packing coloring of period 80 (compared with the 12-packing coloring of period 1028):

$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,10,1,4,1,2,1,3,1,5,1,11,1,2,1,3,1,8,1,9,$
$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,12,1,4,1,2,1,3,1,5,1,13,1,2,1,3,1,9,1,8.$

We now turn our attention on other distance graphs with two chords, i.e. graphs of type $D(a,b)$, with $2 \leq a \leq b$. The smallest example is $D(2,3)$ which is a subgraph of $D(1,2,3) = P^3_\infty$, thus $\chi_\rho(D(2,3)) \leq \chi_\rho(P^3_\infty) \leq 23$. In fact, we show that the upper bound is much less than 22:

Proposition 11.

$$11 \leq \chi_\rho(D(2,3)) \leq 13;$$
$$\chi_\rho(D(2,4)) = 8;$$
$$14 \leq \chi_\rho(D(2,5)) \leq 23;$$
$$\chi_\rho(D(2,6)) = 9.$$

Proof. The graph $D(2,4)$ ($D(2,6)$, respectively) is not connected and consists in two copies of $D(1,2)$ ($D(1,3)$, respectively). Thus $\chi_\rho(D(2,4)) = \chi_\rho(D(1,2)) = 8$ and $\chi_\rho(D(2,6)) = \chi_\rho(D(1,3)) = 9$.

The lower bound $11 \leq \chi_\rho(D(2,3))$ is obtained by calculating the maximum density ρ_i of a color i: it can be seen that $\rho_1 = \frac{2}{5}$ and $\rho_i = \frac{1}{3i+1}$ for $i \geq 2$ and that $\min\{c,\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{3i+1} \geq 1\} = 11$.

For the lower bound $14 \leq \chi_\rho(D(2,5))$, it can be seen that $\rho_1 = \frac{3}{7}$ and $\rho_i = \frac{1}{5i-4}$ for $i \geq 2$ and that $\min\{c,\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{5i-4} \geq 1\} = 14$.

The upper bound $\chi_\rho(D(2,3)) \leq 13$ comes from the 13-packing coloring of period 240 given in Appendix B3 and the bound $\chi_\rho(D(2,5)) \leq 23$ comes from the 23-packing coloring of period 336 given in Appendix B4. □

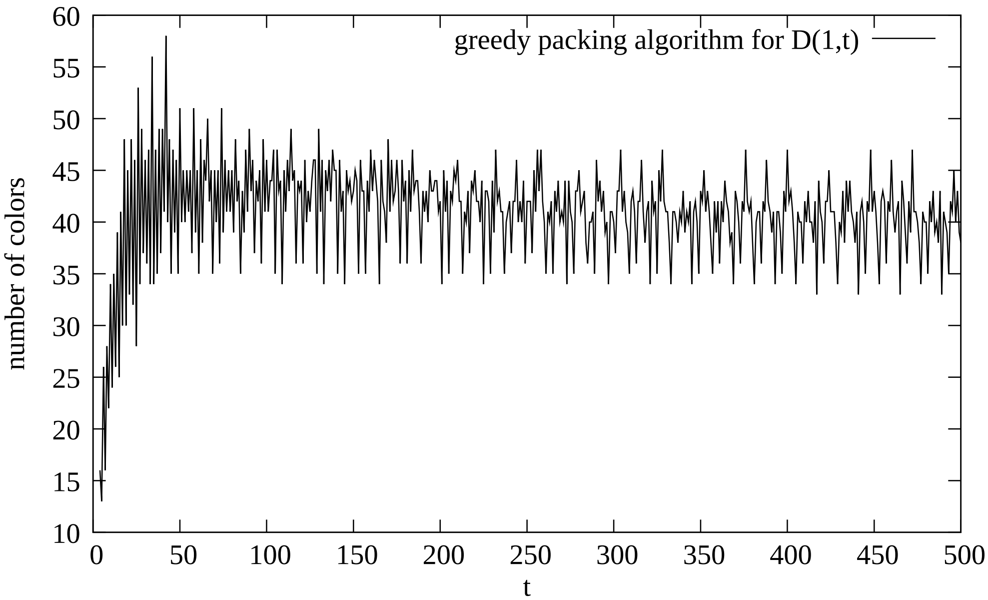

Figure 5: Number of colors for a packing coloring of $D(1,t)$ using a greedy algorithm.

5 Concluding remarks

We have shown that the packing chromatic number of any infinite distance graph with finite D is finite and is at most 40 (81, respectively) for $D = \{1, t\}$ with t being an odd (even, respectively) integer greater than or equal to 447.

Among the many research directions in this area, one can try to find better bounds and/or more simple methods for $D(1,t)$. In fact, running a simple greedy packing coloring algorithm that consists in coloring vertices of a distance graph one-by-one from the left to the right with the smallest color with respect to the constraint, suggests that the upper bounds found in Section 3 can be strengthened. Figure 5 shows the number of colors used by the greedy algorithm for a packing coloring of $D_n(1,t)$ (with $n = 1000000$) as a function of t for the first 500 values of t. One can see on the figure that for large t, the algorithm finds a packing coloring, using between 30 and 50 colors. Moreover, more colors are needed in general when t is even compared to when t is odd. But surprisingly, even if we look only at even (or odd) values of t, the function is not monotonic. We wonder if the same goes for χ_ρ. An interesting future work would be to study more in details the behavior of this greedy algorithm.

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Appendix A: Modification of the color patterns depending on the residues of s modulo 48

For $s \equiv 0$ or $4 \bmod 48$, the color patterns are given in proof of Proposition 6, so only the modifications to obtain the color patterns for other residues of s are presented here.

$s \bmod 48$	8	12	16
P_2	$+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	$+4,5,8,4,5,9$	idem $+\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_2'	$+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	$+4,5,10,4,5,11$	idem $+\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$
P_3	$+6,7,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}32,35,38,41\\33,36,39,42\\34,37,40,43\end{smallmatrix}\right.$	$+6,7,12,13,6,7,14,15$
P_3'	$+6,7,\left\{\begin{smallmatrix}51,53\\52,54\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}44,46,48,50\\45,47,49,51\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}44,46\\45,47\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}48,50\\49,51\end{smallmatrix}\right.$
P_3''	$+6,7,\left\{\begin{smallmatrix}55,56\\57,58\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}52,54,56,58\\53,55,57,59\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}52,54\\53,55\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}56,58\\57,59\end{smallmatrix}\right.$

$s \bmod 48$	20	24
P_2	idem $s \equiv 12$ $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	$+(4,5,8,4,5,9)^2$
P_2'	idem $s \equiv 12$ $+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	$+(4,5,10,4,5,11)^2$
P_3	idem $+\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$	idem $s \equiv 16$ $+6,7,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_3'	$+6,7,\left\{\begin{smallmatrix}51,53\\52,54\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}55,57,59,61\\56,58,60,62\end{smallmatrix}\right.$	0
P_3''	$+6,7,\left\{\begin{smallmatrix}63,66\\64,67\\65,68\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}69,72,75,78\\70,73,76,79\\71,74,77,80\end{smallmatrix}\right.$	$+6,7,22,23,6,7,\left\{\begin{smallmatrix}38,40\\39,41\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}42,44\\43,45\end{smallmatrix}\right.$

$s \bmod 48$	28	32	36
P_2	idem $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	idem $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	$+(4,5,8,4,5,9)^3$
P_2'	idem $+\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	idem $s \equiv 24$ $+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	$+(4,5,10,4,5,11)^3$
P_3	$+6,7,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47,51,55\\48,52,56\\49,53,57\\50,54,58\end{smallmatrix}\right.$	$+(6,7,12,13,6,7,14,15)^2$	idem $+\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_3'	idem $+\left\{\begin{smallmatrix}59,61\\60,62\end{smallmatrix}\right.$	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}44,46\\45,47\end{smallmatrix}\right.$	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}38,40,42,44\\39,41,43,45\end{smallmatrix}\right.$
P_3''	idem $+\left\{\begin{smallmatrix}63,66\\64,67\\65,68\end{smallmatrix}\right.$	0	idem $+\left\{\begin{smallmatrix}46,48\\47,49\end{smallmatrix}\right.$

$s \bmod 48$	40	44
P_2	idem $+\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	idem $s \equiv 36$ $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_2'	idem $s \equiv 32$ $+\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	idem $s \equiv 36$ $+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$
P_3	idem $+6,7,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$	idem $+\left\{\begin{smallmatrix}51,55\\52,56\\53,57\\54,58\end{smallmatrix}\right.$
P_3'	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}51,53\\52,54\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}55,57\\56,58\end{smallmatrix}\right.$	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}59,61\\60,62\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}63,66,69,72\\64,67,70,73\\65,68,71,74\end{smallmatrix}\right.$
P_3''	idem $s \equiv 32$ $+6,7,\left\{\begin{smallmatrix}59,61\\60,62\end{smallmatrix}\right.$	idem $s \equiv 32$ $+6,7,\left\{\begin{smallmatrix}75,78,81,84\\76,79,82,85\\77,80,83,86\end{smallmatrix}\right.$

Appendix B1: A periodic 23-packing coloring of P_∞^3 of period 768

23, 1, 4, 5, 3, 1, 2, 6, 7, 1, 9, 10, 12, 1, 2, 3, 4, 1, 8, 5, 13, 1, 2, 14, 16, 1, 3, 6, 11, 1, 2, 4, 7, 1, 15, 5, 3, 1, 2, 9, 18, 1, 10, 8, 4,
1, 2, 3, 6, 1, 12, 5, 17, 1, 2, 7, 19, 1, 3, 4, 13, 1, 2, 11, 20, 1, 14, 5, 3, 1, 2, 6, 4, 1, 8, 9, 10, 1, 2, 3, 7, 1, 15, 5, 16, 1, 2, 4, 12, 1,
3, 6, 21, 1, 2, 18, 22, 1, 11, 5, 3, 1, 2, 4, 7, 1, 8, 9, 10, 1, 2, 3, 6, 1, 13, 5, 4, 1, 2, 14, 17, 1, 3, 19, 23, 1, 2, 7, 12, 1, 4, 5, 3, 1, 2,
6, 8, 1, 9, 10, 11, 1, 2, 3, 4, 1, 15, 5, 16, 1, 2, 7, 18, 1, 3, 6, 13, 1, 2, 4, 20, 1, 8, 5, 3, 1, 2, 9, 12, 1, 10, 14, 4, 1, 2, 3, 6, 1, 7, 5,
11, 1, 2, 17, 19, 1, 3, 4, 8, 1, 2, 21, 15, 1, 22, 5, 3, 1, 2, 6, 4, 1, 7, 9, 10, 1, 2, 3, 12, 1, 13, 5, 16, 1, 2, 4, 8, 1, 3, 6, 11, 1, 2, 14,
7, 1, 18, 5, 3, 1, 2, 4, 9, 1, 20, 10, 17, 1, 2, 3, 6, 1, 8, 5, 4, 1, 2, 7, 12, 1, 3, 13, 15, 1, 2, 11, 19, 1, 4, 5, 3, 1, 2, 6, 9, 1, 10, 8, 14,
1, 2, 3, 4, 1, 7, 5, 16, 1, 2, 21, 22, 1, 3, 6, 18, 1, 2, 4, 12, 1, 11, 5, 3, 1, 2, 8, 7, 1, 9, 10, 4, 1, 2, 3, 6, 1, 13, 5, 15, 1, 2, 14, 17, 1,
3, 4, 19, 1, 2, 7, 8, 1, 20, 5, 3, 1, 2, 6, 4, 1, 9, 10, 11, 1, 2, 3, 12, 1, 16, 5, 18, 1, 2, 4, 7, 1, 3, 6, 8, 1, 2, 13, 21, 1, 14, 5, 3, 1, 2,
4, 9, 1, 10, 15, 17, 1, 2, 3, 6, 1, 7, 5, 4, 1, 2, 8, 11, 1, 3, 12, 19, 1, 2, 20, 22, 1, 4, 5, 3, 1, 2, 6, 7, 1, 9, 10, 13, 1, 2, 3, 4, 1, 8, 5,
14, 1, 2, 16, 18, 1, 3, 6, 11, 1, 2, 4, 7, 1, 12, 5, 3, 1, 2, 9, 15, 1, 10, 8, 4, 1, 2, 3, 6, 1, 17, 5, 13, 1, 2, 7, 19, 1, 3, 4, 20, 1, 2, 11,
14, 1, 21, 5, 3, 1, 2, 6, 4, 1, 8, 9, 10, 1, 2, 3, 7, 1, 12, 5, 16, 1, 2, 4, 15, 1, 3, 6, 13, 1, 2, 18, 22, 1, 11, 5, 3, 1, 2, 4, 7, 1, 8, 9, 10,
1, 2, 3, 6, 1, 14, 5, 4, 1, 2, 12, 17, 1, 3, 19, 20, 1, 2, 7, 23, 1, 4, 5, 3, 1, 2, 6, 8, 1, 9, 10, 11, 1, 2, 3, 4, 1, 13, 5, 15, 1, 2, 7, 16, 1,
3, 6, 12, 1, 2, 4, 14, 1, 8, 5, 3, 1, 2, 9, 18, 1, 10, 21, 4, 1, 2, 3, 6, 1, 7, 5, 11, 1, 2, 17, 19, 1, 3, 4, 8, 1, 2, 13, 20, 1, 12, 5, 3, 1, 2,
6, 4, 1, 7, 9, 10, 1, 2, 3, 14, 1, 15, 5, 16, 1, 2, 4, 8, 1, 3, 6, 11, 1, 2, 18, 7, 1, 22, 5, 3, 1, 2, 4, 9, 1, 12, 10, 13, 1, 2, 3, 6, 1, 8, 5,
4, 1, 2, 7, 17, 1, 3, 14, 19, 1, 2, 11, 15, 1, 4, 5, 3, 1, 2, 6, 9, 1, 10, 8, 16, 1, 2, 3, 4, 1, 7, 5, 12, 1, 2, 13, 18, 1, 3, 6, 20, 1, 2, 4, 21,
1, 11, 5, 3, 1, 2, 8, 7, 1, 9, 10, 4, 1, 2, 3, 6, 1, 14, 5, 15, 1, 2, 17, 19, 1, 3, 4, 12, 1, 2, 7, 8, 1, 13, 5, 3, 1, 2, 6, 4, 1, 9, 10, 11, 1, 2,
3, 16, 1, 18, 5, 22, 1, 2, 4, 7, 1, 3, 6, 8, 1, 2, 14, 20, 1, 12, 5, 3, 1, 2, 4, 9, 1, 10, 13, 15, 1, 2, 3, 6, 1, 7, 5, 4, 1, 2, 8, 11, 1, 3, 17,
19, 1, 2, 21

Appendix B2: A periodic 16-packing coloring of $D(1,4)$ of period 320

1, 2, 1, 3, 4, 1, 5, 1, 2, 7, 1, 6, 1, 3, 2, 1, 8, 1, 4, 10, 1, 2, 1, 3, 5, 1, 9, 1, 2, 12, 1, 13, 1, 3, 2, 1, 4, 1, 6, 7, 1, 2, 1, 3, 11, 1, 5, 1, 2,
8, 1, 4, 1, 3, 2, 1, 14, 1, 10, 15, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 7, 1, 3, 2, 1, 9, 1, 12, 8, 1, 2, 1, 3, 4, 1, 5, 1, 2, 11, 1, 6, 1, 3, 2, 1, 10,
1, 4, 13, 1, 2, 1, 3, 5, 1, 7, 1, 2, 8, 1, 9, 1, 3, 2, 1, 4, 1, 6, 14, 1, 2, 1, 3, 12, 1, 5, 1, 2, 15, 1, 4, 1, 3, 2, 1, 7, 1, 10, 8, 1, 2, 1, 3, 5, 1,
4, 1, 2, 6, 1, 9, 1, 3, 2, 1, 11, 1, 13, 16, 1, 2, 1, 3, 4, 1, 5, 1, 2, 7, 1, 6, 1, 3, 2, 1, 8, 1, 4, 10, 1, 2, 1, 3, 5, 1, 9, 1, 2, 12, 1, 14, 1, 3,
2, 1, 4, 1, 6, 7, 1, 2, 1, 3, 11, 1, 5, 1, 2, 8, 1, 4, 1, 3, 2, 1, 13, 1, 10, 15, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 7, 1, 3, 2, 1, 9, 1, 12, 8, 1, 2, 1,
3, 4, 1, 5, 1, 2, 11, 1, 6, 1, 3, 2, 1, 10, 1, 4, 14, 1, 2, 1, 3, 5, 1, 7, 1, 2, 8, 1, 9, 1, 3, 2, 1, 4, 1, 6, 13, 1, 2, 1, 3, 12, 1, 5, 1, 2, 15, 1, 4,
1, 3, 2, 1, 7, 1, 10, 8, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 9, 1, 3, 2, 1, 11, 1, 14, 16

Appendix B3 : A periodic 13-packing coloring of $D(2,3)$ of period 240

1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 13, 1, 1, 2, 4, 11, 1, 1, 7, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 8, 10, 2, 1, 1, 12, 3, 5, 1, 1, 2,
4, 6, 1, 1, 7, 3, 2, 1, 1, 9, 11, 13, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 7, 1, 1, 2, 4, 10, 1, 1, 12, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1,
1, 7, 8, 2, 1, 1, 11, 3, 5, 1, 1, 2, 4, 6, 1, 1, 10, 3, 2, 1, 1, 9, 13, 7, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 12, 1, 1, 2, 4, 11, 1, 1, 7, 3,
2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 8, 10, 2, 1, 1, 13, 3, 5, 1, 1, 2, 4, 6, 1, 1, 7, 3, 2, 1, 1, 9, 11, 12, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1,
8, 3, 7, 1, 1, 2, 4, 10, 1, 1, 13, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 7, 8, 2, 1, 1, 11, 3, 5, 1, 1, 2, 4, 6, 1, 1, 10, 3, 2, 1, 1, 9, 12, 7

Appendix B4 : A periodic 23-packing coloring of $D(2,5)$ of period 336

1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 7, 8, 1, 1, 2, 2, 1, 3, 10, 1, 1, 11, 4, 1, 15, 12, 1, 1, 2, 2, 1, 3, 16, 1, 1, 5, 6, 1, 4, 9, 1, 1, 2, 2, 1, 3,
7, 1, 1, 8, 14, 1, 17, 13, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 19, 1, 1, 2, 2, 1, 3, 11, 1, 1, 7, 4, 1, 9, 12, 1, 1, 2, 2, 1, 3, 8, 1, 1, 5, 6,
1, 4, 15, 1, 1, 2, 2, 1, 3, 18, 1, 1, 20, 21, 1, 7, 22, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 9, 1, 1, 2, 2, 1, 3, 8, 1, 1, 11, 4, 1, 13, 12, 1, 1,
2, 2, 1, 3, 7, 1, 1, 5, 6, 1, 4, 14, 1, 1, 2, 2, 1, 3, 16, 1, 1, 17, 23, 1, 9, 19, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 7, 8, 1, 1, 2, 2, 1, 3, 10, 1

, 1, 11, 4, 1, 15, 12, 1, 1, 2, 2, 1, 3, 13, 1, 1, 5, 6, 1, 4, 9, 1, 1, 2, 2, 1, 3, 7, 1, 1, 8, 18, 1, 14, 20, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 21, 1, 1, 2, 2, 1, 3, 11, 1, 1, 7, 4, 1, 9, 12, 1, 1, 2, 2, 1, 3, 8, 1, 1, 5, 6, 1, 4, 13, 1, 1, 2, 2, 1, 3, 15, 1, 1, 16, 17, 1, 7, 19, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 9, 1, 1, 2, 2, 1, 3, 8, 1, 1, 11, 4, 1, 14, 12, 1, 1, 2, 2, 1, 3, 7, 1, 1, 5, 6, 1, 4, 18, 1, 1, 2, 2, 1, 3, 13, 1, 1, 20, 22, 1, 9, 23

On Packing Colorings of Distance Graphs

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Abstract

The *packing chromatic number* $\chi_\rho(G)$ of a graph G is the least integer k for which there exists a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices of color i are at distance at least $i+1$. This paper studies the packing chromatic number of infinite distance graphs $G(\mathbb{Z}, D)$, i.e. graphs with the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| \in D$. We present lower and upper bounds for $\chi_\rho(G(\mathbb{Z}, D))$, showing that for finite D, the packing chromatic number is finite. Our main result concerns distance graphs with $D = \{1, t\}$ for which we prove some upper bounds on their packing chromatic numbers, the smaller ones being for $t \geq 447$: $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 40$ if t is odd and $\chi_\rho(G(\mathbb{Z}, \{1, t\})) \leq 81$ if t is even.

Keywords: graph coloring; packing chromatic number; distance graph.

1 Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A *packing k-coloring* of a graph G is a mapping f from $V(G)$ to $\{1, 2, \ldots, k\}$ such that any two vertices of color i are at distance at least $i+1$ (thus vertices of color i form an i-packing of G). The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k for which G has a packing k-coloring.

This parameter was introduced recently by Goddard et al. [?] under the name of *broadcast chromatic number* and the authors showed that deciding whether $\chi_\rho(G) \leq 4$ is NP-hard. Fiala and Golovach [?] showed that the problem remains NP-complete for trees. Brešar et al. [?] studied the problem on Cartesian products graphs, hexagonal lattice and trees, using the name of packing chromatic number. Other studies on this parameter mainly concern infinite graphs, with a natural question to be answered : *is a given infinite graph has finite packing chromatic number ?* Goddard et al. answered by the positive for the infinite two dimensional square grid by showing $9 \leq \chi_\rho \leq 23$. The lower bound was later improved to 10 by Fiala et al. [?] and then to 12 by Ekstein et al. [?]. The upper bound was recently improved by Holub and Soukal [?] to 17. Fiala et al. [?] showed that the infinite hexagonal grid has packing chromatic number 7; while the infinite triangular lattice along with the 3-dimensional square lattice was shown to admit no finite packing coloring by Finbow and Rall [?]. Infinite product graphs were considered by Fiala et al. [?] that showed that the product of a finite path (of

order at least two) by the 2-dimensional square grid has infinite packing chromatic number while the product of the infinite path by any finite graph has finite packing chromatic number.

The (infinite) *distance graph* $G(\mathbb{Z}, D)$ with distance set $D = \{d_1, d_2, \ldots, d_k\}$, where d_i being positive integers, has the set $\mathbb{Z}$ of integers as vertex set, with two distinct vertices $i, j \in \mathbb{Z}$ being adjacent if and only if $|i - j| \in D$. The *finite distance graph* $G_n(D)$ is the subgraph of $G(\mathbb{Z}, D)$ induced by vertices $0, 1, \ldots, n - 1$. An edge between vertices a and $a + d_i$ will be called a d_i-edge.

The study of distance graphs was initiated by Eggleton et al. [**?**]. A large amount of work has focused on colorings of distance graphs [**?**, **?**, **?**, **?**, **?**, **?**], but other parameters have also been studied on distance graphs, like the feedback vertex set problem [**?**].

The aim of this paper is to study the packing chromatic number of infinite distance graphs, with particular emphasis on the case $D = \{1, t\}$. In section 2, we bound the packing chromatic number of the infinite path power (i.e. infinite distance graph with $D = \{1, 2, \ldots, t\}$). Section 3 concerns packing colorings of distance graphs with $D = \{1, t\}$, for which we prove some lower and upper bounds on the number of colors. Exact or sharp results for the packing chromatic number of some other 4-regular distance graphs are presented in Section 4. Section 5 concludes the paper with some remarks and open questions.

Our results about the packing chromatic number of $G(\mathbb{Z}, D)$ for some small values of D (from Sections 2 and 4) are summarized in Table 1. The source code (`C++`) of the program used to obtain most of the lower bounds and some upper bounds along with long sequences of colors not given in this paper can be found at [**?**].

D	$\chi_\rho \geq$	$\chi_\rho \leq$	period
$1,2$	8^*	8	54
$1,3$	9^*	9	32
$1,4$	11	16	320
$1,5$	10^*	12	1028
$1,6$	11^*	23	1917
$1,7$	10^*	15	640
$1,8$	11^*	25	5184
$1,9$	10^*	18	576
$1,2,3$	19	23	768
$2,3$	11	13	240
$2,4$	8^*	8	54
$2,5$	14	23	336
$2,6$	9^*	9	32

Table 1: Lower and upper bounds for the packing chromatic number of $G(\mathbb{Z}, D)$ for different values of D. In the fourth column are the periods of the colorings giving the upper bounds. (*: bound obtained by a computer search)

The bounds of Section 3 are summarized in the following Proposition:

Proposition 1.

$$\chi_\rho(G(\mathbb{Z},\{1,t\})) \leq \begin{cases} 86, & t = 2q+1, q \geq 36 \\ 40, & t = 2q+1, q \geq 223 \\ 173, & t = 2q, q \geq 87 \\ 81, & t = 2q, q \geq 224 \\ 29, & t = 96q \pm 1, q \geq 1 \\ 59, & t = 96q + 1 \pm 1, q \geq 1 \end{cases}$$

2 Path Powers

Let $D^t = G(\mathbb{Z}, \{1, 2, \ldots, t\})$ be the t^{th} power of the two-ways infinite path and let $P_n^t = G_n(\{1, 2, \ldots, t\})$ be the t^{th} power of the path P_n on n vertices.

We first present an asymptotic result on the packing chromatic number:

Proposition 2. *$\chi_\rho(D^t) = (1 + o(1))3^t$ and $\chi_\rho(D^t) = \Omega(e^t)$.*

Proof. D^t is a spanning subgraph of the lexicographic product $\mathbb{Z} \circ K_t$ (see Figure 1). Then, as Godard et al. [**?**] showed that $\chi_\rho(\mathbb{Z} \circ K_t) = (1 + o(1))3^t$, the same upper bound holds for D^t. To prove the lower bound, since two vertices of color i have to be $it + 1$ apart, then for any packing coloring of D^t using at most c colors, c must satisfy:

$$\sum_{i=1}^{c} \frac{1}{it+1} \geq 1.$$

Since

$$\sum_{i=1}^{c} \frac{1}{it+1} < \sum_{i=1}^{c} \frac{1}{it} = \frac{1}{t}\sum_{i=1}^{c} \frac{1}{i} = \frac{H_c}{t},$$

where H_n is the nth harmonic number and since $H_n = \Omega(\ln(n))$, then $\frac{H_c}{t} \geq 1$ implies $c = \Omega(e^t)$.

□

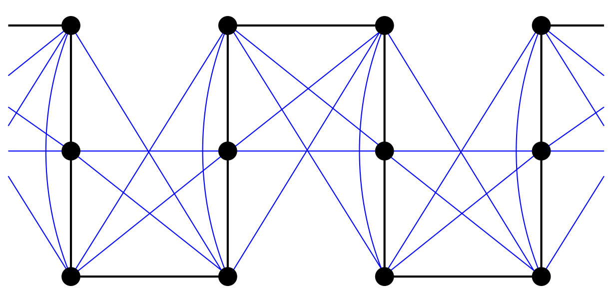

Figure 1: The infinite distance graph D^3 as a subgraph of the lexicographic product $\mathbb{Z} \circ K_3$.

Corollary 1. *For any finite subset D of $\mathbb{N}$, the packing chromatic number of $G(\mathbb{Z}, D)$ is finite.*

For very small t, exact values or sharp bounds for the packing chromatic number can be calculated:

Proposition 3.

$$\chi_\rho(D^2) = 8.$$

Proof. A periodic packing coloring of period 54 using 8 colors is given by the following sequence of length 54 :

$$8,1,2,6,1,4,3,2,1,5,7,1,2,3,4,1,6,2,1,8,3,1,2,4,1,5,7,$$
$$1,3,2,1,6,4,1,2,3,1,8,5,1,2,4,1,3,6,1,2,7,1,5,4,2,1,3.$$

One can check that the distance (considered cyclically) between two occurrences of a color i in this sequence is at least $2i+1$, hence this coloring is a packing coloring of D^2.

Moreover, by checking all cases with the help of a computer, we find that 7 colors are not sufficient for a packing coloring of P_{26}^2. □

Proposition 4.

$$19 \le \chi_\rho(D^3) \le 23.$$

Proof. The upper bound comes from a coloring of period 768 using 23 colors described by the sequence of length 768 of Appendix B1.

To prove the lower bound, we consider the maximum density ρ_i of a color i in a packing coloring of D^3. As $d(j,k) = \lceil \frac{k-j}{3} \rceil$, then $\rho_i = \frac{1}{3i+1}$. However, for the colors 1 and 2, we show that only at most 10 over 28 consecutive vertices of D^3 can be colored with these colors: at most $28/4 = 7$ vertices can be colored 1 and at most $28/7 = 4$ vertices can be colored 2, but as $\mathrm{lcm}(4,7) = 28$, we have to choose between color 1 and color 2 for at least one vertex, thus at most $7+4-1 = 10$ vertices can be colored 1 or 2. Then, an easy computation gives that $\chi_\rho(D^3) \ge \min\{c,\ \frac{10}{28} + \sum_{i=3}^{c} \frac{1}{3i+1} \ge 1\} = 19$. □

3 Distance graphs with $D = \{1, t\}$

Let $D(a,b) = G(\mathbb{Z}, \{a,b\})$ be the infinite distance graph with chord lengths a and b. Let also $D_n(a,b) = G_n(\{a,b\})$.

The case $a = 1$ and $b = 2$ was discussed in the previous section, so we now consider $D(1,t)$ with $t \ge 3$.

The general method we shall use will be to cut the graph into blocks B_i of size $s = t-1$ or $s = t+1$, depending on the value of t and to color each block by a predefined color pattern. Figure 2 illustrate the grid-like structure of $D(1,t)$.

Let $s = t \pm 1$ and let $B_i = \{is, is+1, \ldots, (i+1)s-1\}$. Then $D(1,t) = \cup_{i=-\infty}^{+\infty} B_i$. Remark that if $t = 4p-1 = s-1$, then each B_i is an induced cycle of $D(1,t)$ of length $s = t+1 = 4p$ (see Figure 2). By a color pattern P, we mean a sequence of s colors $(c_1, c_2, \ldots, c_s)$ (that will be associated to a block B_i). If S is a sequence of integers, S^p is the sequence obtained by repeating p times S.

We first need to know the distance between two vertices in $D(1,t)$.

Lemma 1. *The distance between two vertices a and b of $D(1,t)$ is $d(a,b) = \min(q+r, q+1+t-r)$, where $|b-a| = qt+r$, with $0 \le r < t$.*

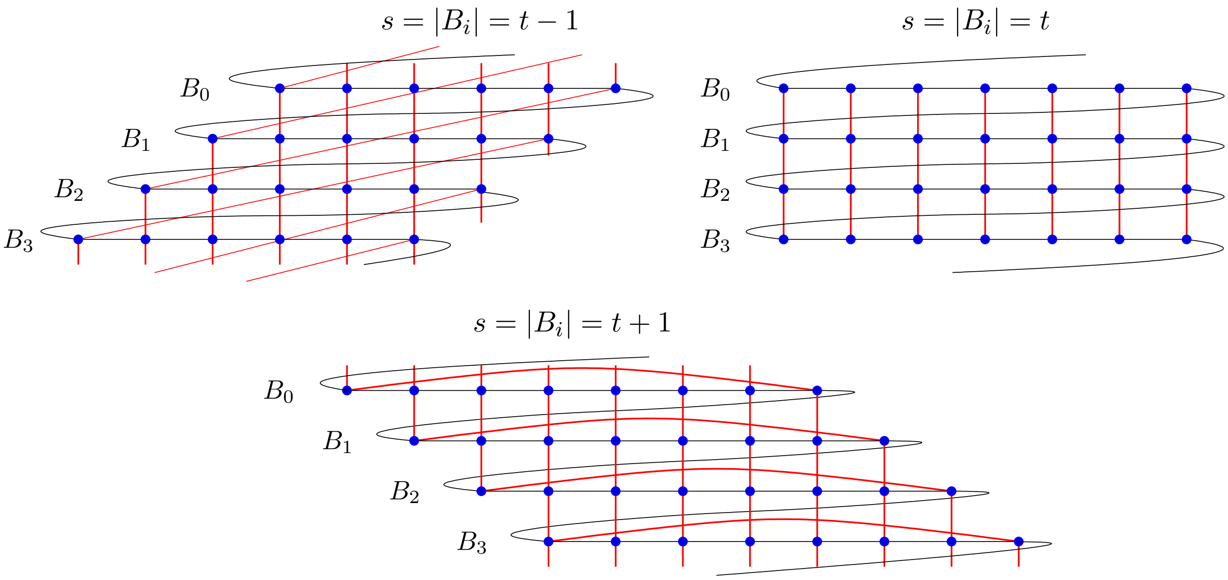

Figure 2: Three block-representations of $D(1,t)$

Proof. Assume w.l.o.g. that $b \geq a$. Any minimal path between a and b uses either q t-edges and r 1-edges or $q+1$ t-edges and $t-r$ 1-edges. □

The key Lemma of our method is the following which indicates when a color of a pattern can be re-used on another pattern.

Lemma 2. *Let $t = 4p-1$ or $4p$, $s = t+1$ and let P, P' be two color patterns containing the color $m < t$ at the same places and no common color greater than m. If P lies on block B_i of $D(1,t)$, then P' can be placed on block B_j whenever $|j-i| > \frac{m}{2}$.*

Proof. Assume without loss of generality that $i = 0$. Let m be a color (an integer) lying on block B_0 (perhaps in several places) and on block B_j (and not on blocks B_h, $0 < h < j$). Let $a_1, a_2, \ldots, a_d$ be the positions of the occurrences of color m in B_0 and let $b_1, b_2, \ldots, b_d$ be the positions of the occurrences of color m in B_j. Then $b_k = a_k + js$.

We are going to show that for any $1 \leq k, \ell \leq d$, we have $d(a_k, b_\ell) > m$ whenever $j > \frac{m}{2}$. For this, we distinguish two cases depending on the values of k and ℓ.

Case 1. $k = \ell$. Then $b_\ell - a_k = js = jt + j$. Hence, by virtue of Lemma 1, $d(a_k, b_\ell) = \min(j+j, j+1+t-j) = \min(2j, t+1) > m$ as soon as $2j > m$, i.e. $j > \frac{m}{2}$.

Case 2. $k \neq \ell$. Let $\delta_b = |b_\ell - b_k|$. Then, as the coloring is a packing coloring of B_j (which is a cycle of length $t+1$), we have

$$(\ell - k)m < \delta_b < t - m + 1 \tag{1}$$

Subcase 2.1. $k < \ell$. Then $b_\ell - a_k = jt + j + \delta_b$.

If $j+\delta_b<t$, then $b_\ell-a_k=jt+j+\delta_b$. Hence, Lemma 1 gives $d(a_k,b_\ell)=\min(j+j+\delta_b,j+1+t-j-\delta_b)=\min(2j+\delta_b,t+1-\delta_b)>m$ since $j>\frac{m}{2}$ and $t+1-\delta_b>m$ by (1).

If $j+\delta_b\geq t$, then $b_\ell-a_k=(j+1)t+(j+\delta_b-t)$. Hence, Lemma 1 gives $d(a_k,b_\ell)=\min(j+1+j+\delta_b-t,j+2+t-j-\delta_b+t)=\min(2j+1+\delta_b-t,2t+2-\delta_b)>m$ since $2j+1+\delta_b-t=j+1+j+\delta_b-t\geq j+1\geq t-\delta_b+1>m$ and $2t+2-\delta_b\geq t+2>m+2$.

Subcase 2.2. $k>\ell$. Then $b_\ell-a_k=js-(b_k-b_\ell)=jt+j-\delta_b$.

If $\delta_b<j$, then $b_\ell-a_k=jt+(j-\delta_b)$. Hence, Lemma 1 gives $d(a_k,b_\ell)=\min(j+j-\delta_b,j+1+t-j+\delta_b)=\min(2j-\delta_b,t+1+\delta_b)>m$ since $j\geq\delta_b>m$ and $t+1+\delta_b>t>m$.

If $\delta_b>j$, then $b_\ell-a_k=(j-1)t+(t+j-\delta_b)$. Hence, Lemma 1 gives $d(a_k,b_\ell)=\min(j-1+t+j-\delta_b,j+t-t-j+\delta_b)=\min(2j+t-1-\delta_b,\delta_b)>m$ since by (1), $t-\delta_b>m-1$ and $\delta_b>m$.

See Figure 3 for an illustration when $t=7$.

□

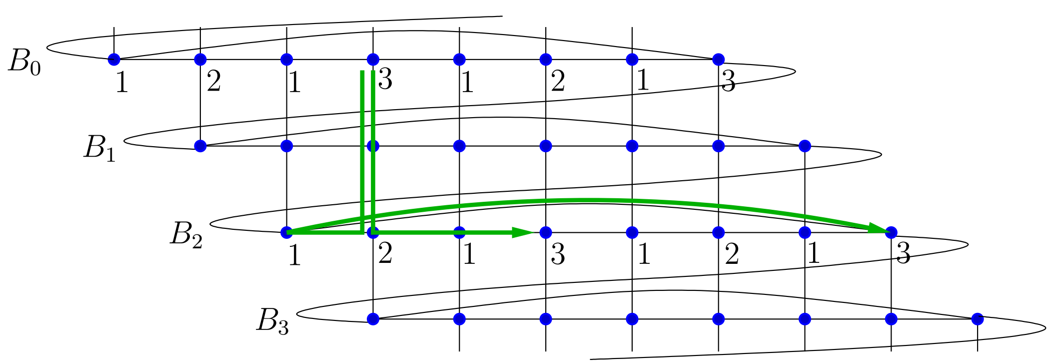

Figure 3: Two shortest paths in $D(1,7)$ between a vertex colored 3 in block B_0 and two other vertices colored 3 in B_2.

The next result shows that a packing coloring of $D(1,t)$ by patterns that satisfies conditions of Lemma 1 is also a packing coloring of $D(1,t+2)$.

Lemma 3. *for $t=4p-1$ or $4p$, any pattern packing coloring of $D(1,t)$ is a packing coloring of $D(1,t+2)$.*

Proof. Let $t=4p-1$ (resp. $4p$) and f be a pattern packing coloring of $D(1,4p-1)$ (resp. $D(1,4p)$). Assume $D(1,4p+1)$ (resp. $D(1,4p+2)$) is now colored by f. Fix $t'=t+2$. Then we have to check the two cases of the proof of Lemma 2 with $m<t=t'-2$ (and $s=t+1=t'-1$).

Case 1. $k=\ell$. Then $b_\ell-a_k=js=jt'-j=(j-1)t'+t'-j$. Hence $d(a_k,b_\ell)=\min(j-1+t'-j,j+t'-t'+j)=\min(t'-1,2j)>m$ since $m<t'-2$ and $2j>m$.

Case 2. $k \neq \ell$. Let $\delta_b = |b_\ell - b_k|$. Since $t = t' - 2$, then Equation (1) now becomes

$$(\ell - k)m < \delta_b < t' - m - 1 \tag{2}$$

Subcase 2.1. $k < \ell$. Then $b_\ell - a_k = jt' - j + \delta_b$.

If $\delta_b - j \geq 0$, then we have $d(a_k, b_\ell) = \min(j - j + \delta_b, j + 1 + t' + j - \delta_b) = \min(\delta_b, 2j + 1 + t' - \delta_b) > m$ since $\delta_b > m$ and $t' - \delta_b > m + 1$ by (2).

If $\delta_b - j < 0$, then $j > \delta_b > m$. In that case, we have $b_\ell - a_k = (j-1)t' + (t' - j + \delta_b)$ and $d(a_k, b_\ell) = \min(j - 1 + t' - j + \delta_b, j + t' - t' + j - \delta_b) = \min(t' - 1 + \delta_b, 2j - \delta_b) > m$ since by hypothesis, $j > \delta_b > m$.

Subcase 2.2. $k > \ell$. Then $b_\ell - a_k = js - \delta_b = jt' - j - \delta_b$.

If $j + \delta_b \leq t'$, then $b_\ell - a_k = (j-1)t' + (t' - j - \delta_b)$ and $d(a_k, b_\ell) = \min(j - 1 + t' - j - \delta_b, j + t' - t' + j + \delta_b) = \min(t' - 1 - \delta_b, 2j + \delta_b) > m$ since $t' - 1 - \delta_b > m$ by (2).

If $j + \delta_b > t'$, as $\delta_b < t' - m - 1$, then $j > m + 1$. We have $b_\ell - a_k = (j-2)t' + (2t' - j - \delta_b)$ and $d(a_k, b_\ell) = \min(j - 2 + 2t' - j - \delta_b, j - 1 + t' - 2t' + j + \delta_b) = \min(2t' - 2 - \delta_b, 2j - t' - 1 + \delta_b) > m$ since $2j - t' - 1 + \delta_b = j + \delta_b - t' + j - 1$ and $j - 1 > m$ in that case.

See Figure 4 for an illustration when $t = 9$. □

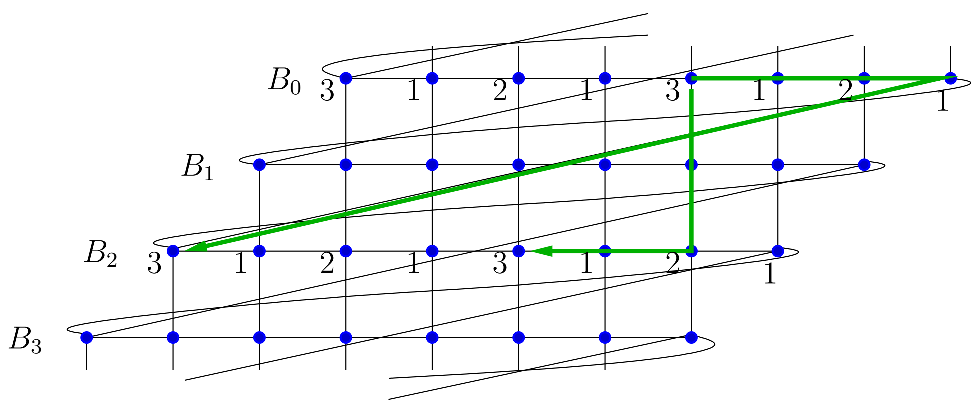

Figure 4: Two shortest paths in $D(1, 9)$ between a vertex colored 3 in block B_0 and two other vertices colored 3 in B_2.

3.1 $D(1, t)$ with odd t

We now present a construction to obtain a packing coloring of $D(1, t)$ for $t \equiv \pm 1 \bmod 96$, with at most 29 colors. To prove this inequality, we shall construct a packing coloring by defining a set of color patterns that will be used cyclically to color the blocks of $D(1, t)$.

Proposition 5. *For $t \equiv -1 \bmod 96$ or $t \equiv 1 \bmod 96$,*

$$\chi_\rho(D(1, t)) \leq 29.$$

Proof. Let $s = 96q$ for some $q \geq 1$ and $t = 96q \pm 1$. We use the following color patterns:

$P_1 = (1, 2, 1, 3)^{24q}$
$P_2 = (1, 4, 1, 5, 1, 8, 1, 4, 1, 5, 1, 9)^{8q}$
$P_2' = (1, 4, 1, 5, 1, 10, 1, 4, 1, 5, 1, 11)^{8q}$
$P_3 = (1, 6, 1, 7, 1, 12, 1, 13, 1, 6, 1, 7, 1, 14, 1, 15)^{6q}$
$P_3' = (1, 6, 1, 7, 1, 16, 1, 17, 1, 6, 1, 7, 1, 18, 1, 19, 1, 6, 1, 7, 1, 20, 1, 21)^{4q}$
$P_3'' = (1, 6, 1, 7, 1, 22, 1, 23, 1, 6, 1, 7, 1, 24, 1, 25, 1, 6, 1, 7, 1, 26, 1, 27, 1, 6, 1, 7, 1, 28, 1, 29)^{3q}$

Then, a packing coloring of $D(1, t)$ using these patterns is constructed by assigning inductively to 16 consecutive blocks B_i the pattern

$$\mathcal{P} = (P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3', P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3'').$$

By virtue of Lemmas 1 and 2, we only have to show that color patterns with common colors are separated enough in $\mathcal{P}$: the distance (considered cyclically) between two patterns P and P' with a maximum common color m in $\mathcal{P}$ has to be greater than $\frac{m}{2}$. It is easily seen that this is effectively the case, as indicated in the following table:

Patterns	Cyclic distance	Max common color
P_1, P_1	2	3
P_2, P_2	8	9
P_2', P_2'	8	11
P_2, P_2'	4	5
P_3, P_3	8	15
P_3', P_3'	16	21
P_3'', P_3''	16	29
P_3, P_3'	4	7
P_3, P_3''	8	7
P_3', P_3''	4	7

Hence, the coloring is a packing coloring of $D(1, t)$ with 29 colors. □

We now generalize this method to color any distance graph $D(1, t)$ with sufficiently large odd t.

Proposition 6. *For any odd $t \geq 73$,*

$$\chi_\rho(D(1, t)) \leq 86.$$

Proof. The method we are going to use to define a packing coloring of $D(1, t)$ with at most 86 colors (the exact number of colors used will vary between 31 and 86, depending on the residue of s modulo 48) is similar with the one of proof of Proposition 5: $D(1, t)$ will be colored by the pattern (of color patterns) $\mathcal{P} = (P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3', P_1, P_2, P_1, P_3, P_1, P_2', P_1, P_3'')$ defined in proof of Proposition 5, where the P_i, P_i' and P_i'' will be defined depending on the residue of s modulo 48.

Our base case is $s \equiv 0 \bmod 48$, *i.e.* $t = 48q \pm 1$, for which we use the following color patterns (which are similar with the color patterns for $t = 96q \pm 1$, except for P_3''):

P_1	$(1,2,1,3)^{12q}$
P_2	$(1,4,1,5,1,8,1,4,1,5,1,9)^{4q}$
P_2'	$(1,4,1,5,1,10,1,4,1,5,1,11)^{4q}$
P_3	$(1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^{3q}$
P_3'	$(1,6,1,7,1,16,1,17,1,6,1,7,1,18,1,19,1,6,1,7,1,20,1,21)^{2q}$
P_3''	$(1,6,1,7,1,22,1,23,1,6,1,7,1,24,1,25,1,6,1,7,1,26,1,27,$ $1,6,1,7,1,22,1,23,1,6,1,7,1,28,1,29,1,6,1,7,1,30,1,31)^{q}$

Now, for $s \equiv 4 \bmod 48$ with $q \geq 2$, we are going to modify the above patterns by adding some new colors (from 32 to 58) at the end of the patterns, as indicated in the following table:

P_1	$(1,2,1,3)^{12q},1,2,1,3$
P_2	$(1,4,1,5,1,8,1,4,1,5,1,9)^{4q},1,\left\{\begin{array}{l}32\\33\\34\end{array}\right.,1,\left\{\begin{array}{l}35\\36\\37\end{array}\right.$
P_2'	$(1,4,1,5,1,10,1,4,1,5,1,11)^{4q},1,\left\{\begin{array}{l}38\\39\\40\end{array}\right.,1,\left\{\begin{array}{l}41\\42\\43\end{array}\right.$
P_3	$(1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^{3q},1,\left\{\begin{array}{l}44\\45\\46\end{array}\right.,1,\left\{\begin{array}{l}47\\48\\49\\50\end{array}\right.$
P_3'	$(1,6,1,7,1,16,1,17,\ldots,1,6,1,7,1,20,1,21)^{2q},1,\left\{\begin{array}{l}51\\52\end{array}\right.,1,\left\{\begin{array}{l}53\\54\end{array}\right.$
P_3''	$(1,6,1,7,1,22,1,23,\ldots,1,6,1,7,1,30,1,31)^{q},1,\left\{\begin{array}{l}55\\56\end{array}\right.,1,\left\{\begin{array}{l}57\\58\end{array}\right.$

For instance for P_2, two new colors from $\{32,33,\ldots,37\}$ are used in turn, i.e. the last four integers of successive occurrences of P_2 will be $1,32,1,35$; $1,33,1,36$; $1,34,1,37$; $1,32,1,35$; $\ldots$. As the period of the pattern P_2 in the coloring is 8, then two patterns ending, say, by $1,34,1,37$ will be repeated each 24 patterns and thus Lemma 2 asserts that two occurrences of the color 34 are separated quite enough and the same goes for the color 37.

As can be seen, seven new colors are used for P_3, since if only six new colors were used, the patterns containing the color 48 (or 49) will be repeated each 24 times, but it is not sufficient to ensure that vertices colored by 48 (or 49) are separated quite enough. Then, the last four integers of successive occurrences of P_3 will be $1,42,1,47$; $1,43,1,48$; $1,44,1,49$; $1,42,1,50$; $1,43,1,47$, ... Therefore, two occurrences of P_3 will end by the same four colors each 12 times and will end by the same color (from $\{47,48,49,50\}$) each 4 times (hence each 32 sequences). With Lemma 2, we know that it will not cause any conflict if the colors used are less than $32*2=64$.

For the other residues of s modulo 48 (with $q \geq 2$ if $s \bmod 48 < 24$), the added colors (to the base case $s \equiv 0 \bmod 48$) are given in the four tables of Appendix A (without the 1s, for sake of brevity). Notice also that for the case $s \equiv 24 \bmod 48$, the colors 22 and 23 are reused to complete the pattern P_3''. □

As the next Proposition shows, increasing the minimum value of t allows to shorten the number of colors for a packing coloring of $D(1,t)$.

Proposition 7. *For any odd $t \geq 447$,*

$$\chi_\rho(D(1,t)) \leq 40.$$

Proof. The main idea to obtain a packing coloring of $D(1,t)$ is to modify the coloring of $D(1,t)$ for $s \equiv 0 \bmod 96$ given in Proof of Proposition 5 by adding only one new color α_i to each block B_i. In order to do that, depending on the value of s, α_i must be placed in several

(quasi evenly distributed) positions in block B_i. The conditions for the coloring to remain a packing coloring are i) that vertices of B_i colored α_i have to be (cyclically) distant quite enough and that ii) the color α_i is not reused in another block B_j with $|j-i| \leq \frac{\alpha_i}{2}$ (necessary condition for Lemma 2).

Color patterns are modified in this way:
$P_1 = (1,2,1,3)^{q_1}, s = 4q_1$
$P_2 = (1,4,1,5,1,8,1,4,1,5,1,9)^{q_2} \sqcup (1, \left\{ \begin{smallmatrix} 32 \\ 33 \\ 34 \end{smallmatrix} \right.)^{r_2}, s = 12q_2 + 2r_2, 0 \leq r_2 \leq 4$
$P'_2 = (1,4,1,5,1,10,1,4,1,5,1,11)^{q_2} \sqcup (1, \left\{ \begin{smallmatrix} 35 \\ 36 \\ 37 \end{smallmatrix} \right.)^{r_2}, s = 12q_2 + 2r_2, 0 \leq r_2 \leq 4$
$P_3 = (1,6,1,7,1,12,1,13,1,6,1,7,1,14,1,15)^{q_3} \sqcup (1, \left\{ \begin{smallmatrix} 38 \\ 39 \\ 40 \end{smallmatrix} \right.)^{r_3}, s = 16q_3 + 2r_3, 0 \leq r_3 \leq 6$
$P'_3 = (1,6,1,7,1,16,1,17,\ldots,1,6,1,7,1,20,1,21)^{q_4} \sqcup (1,30)^{r_4}, s = 24q_4 + 2r_4, 0 \leq r_4 \leq 10$
$P''_3 = (1,6,1,7,1,22,1,23,\ldots,1,6,1,7,1,28,1,29)^{q_5} \sqcup (1,31)^{r_5}, s = 32q_5 + 2r_5, 0 \leq r_5 \leq 14$,
where $S \sqcup (1,\alpha)^r$ is a sequence obtained by inserting r quasi evenly cyclically distributed occurrences of the pair $(1,\alpha)$ in the sequence S (insertions are possible only after a color > 1, in order to keep the sequence alternate between color 1 and other colors).

For example, $(1,4,1,5,1,8,1,4,1,5,1,9)^3 \sqcup (1,\alpha)^5$ can be rewritten as
$(1,4,1,5,1,8,1,\alpha,1,4,1,5,1,9,1,\alpha,1,4,1,5,1,8,1,4,1,\alpha,1,5,1,9,1,4,1,5,1,\alpha,1,8,1,4,1,5,1,9,1,\alpha)$.

In order to satisfy Condition i) and as the pairs $(1,\alpha)$ have to be inserted only on even positions, we must have $2\lfloor\lfloor\frac{|S|}{r}\rfloor/2\rfloor \geq \alpha$. Hence the worst case for this separation constraint is for the color 31 in P''_3 when $r_5 = 14$: one can insert 14 occurrences of $(1,31)$ if $2\lfloor\lfloor\frac{32q_5}{14}\rfloor/2\rfloor \geq 31$, which is true as soon as $q_5 = 14$ and thus $s = 448$.

Moreover, it can be seen that the added color in each pattern is chosen in such a way that Condition ii) is satisfied. For P_2, colors 32, 33 and 34 will be used in turn (i.e. the first block colored by P_2 will use color 32, the second 33, the third 34 and so on... And the same goes for P'_2 and P_3. The patterns P'_3 (P''_3, respectively) are distant quite enough in $\mathcal{P}$ to use always the same new color (30 and 31, respectively).

□

Remark that the above method can produce a packing coloring using less than 40 colors, depending on the value of s (i.e. if some r_i are equal to zero). Notice also that combining the methods of Proposition 6 and 7 allows to define a packing coloring for $95 \leq t \leq 447$ using a number of colors lying between 40 and 86.

3.2 $D(1,t)$ with even t

In this subsection, we adapt the method of the previous subsection to obtain upper bounds for the packing chromatic number of $D(1,t)$ when t is even. Although the main idea is the same, it is more complicated (and much more colors are needed) than for the odd case because of the fact that one cannot alternate between color one and other colors too many times (at most $t/2$ times).

The distance graph $D(1,t)$, with $t = 4p$ or $4p+2$ is cut in blocks $B_0, B_1, \ldots$ of size $s = 4p+1$ and new color patterns are constructed by inserting a new color at the end of each pattern (of length $s' = s-1$) of Proofs of Propositions 5, 6 and 7.

Proposition 8. *For any even t,*

- *if $t \geq 174$, then $\chi_\rho(D(1,t)) \leq 173$;*

- *if* $t \geq 448$, *then* $\chi_\rho(D(1,t)) \leq 81$;
- *if* $t \equiv 0$ *or* $2 \bmod 96$, *then* $\chi_\rho(D(1,t)) \leq 59$.

Proof. By Lemmas 2 and 3 and by Propositions 5, 6 and 7, it remains to add the missing color in each color pattern. This problem is equivalent to the one of coloring the infinite path P_∞ with colors from $\{k_1, k_1+1, \ldots, k_2\}$ such that vertices of color i are at distance greater than $\frac{i}{2}$ (and with $t > k_2$).

We are going to show, by induction on k_1, that $k_2 \leq 2k_1 - 1$. For $k_1 = 2$, vertices can be colored by alternating color 2 and color 3, so $k_2 = 3$. Assume that P_∞ can be colored with colors from $\{k_1, k_1+1, \ldots, k_2 \leq 2k_1 - 1\}$ and let $k_1' = k_1 + 1$. Replace now color k_1 by colors k_2+1 and k_2+2 alternatively. Then the maximum color used is $k_2' = k_2+2 \leq 2k_1+1 = 2k_1'-1$ and the constraint is satisfied since if vertices x and y are colored k_2+2 then their mutual distance satisfies $d(x,y) > 2\frac{k_1}{2} \geq \frac{k_2+1}{2} > \frac{k_2}{2}$.

As the colorings defined in Proof of Proposition 6 (Propositions 7 and 5, respectively) use at most 86 (40 and 29, respectively) colors, then we obtain a packing coloring of $D(1,2t)$ with at most $2 \times 87 - 1 = 173$ colors (81 and 59, respectively).

□

In fact, it seems that less than $2k_1 - 1$ colors are enough for such a coloring. When $k_1 = 87$, a computation gives $k_2 = 152$ for such a coloring; when $k_1 = 41$, we find $k_2 = 71$ and when $k_1 = 30$, we find $k_2 = 53$.

4 $D(a,b)$ with small a and b

Results of Section 3 do not apply for $D(1,t)$ with small t, however it is possible to derive exact or sharp results for such graphs, using ad-hoc methods.

Proposition 9.

$$\chi_\rho(D(1,3)) = 9.$$

Proof. first, remark that the graph-distance $d(i,j)$ between vertex i and vertex $j \geq i$ is $d(i,j) = \lfloor \frac{j-i}{3} \rfloor + (j-i) \bmod 3$.

A 13-packing coloring of $D(1,3)$ of period 32 is given by the following sequence:
$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,4,1,5,1,2,1,3,1,8,1,9$.

It is routine to check that vertices of a same color are distant quite enough.

Again, with the help of a computer, we find that 8 colors are not sufficient for a packing coloring of $D_{100}(1,3)$.

□

Proposition 10.

$$11 \leq \chi_\rho(D(1,4)) \leq 16;$$

$$10 \leq \chi_\rho(D(1,5)) \leq 12;$$

$$11 \leq \chi_\rho(D(1,6)) \leq 23;$$

$$10 \leq \chi_\rho(D(1,7)) \leq 15;$$

$$11 \leq \chi_\rho(D(1,8)) \leq 25;$$

$$10 \leq \chi_\rho(D(1,9)) \leq 18.$$

Proof. For $D(1,4)$, the lower bound is obtained by calculating the maximum density ρ_i of a color i: it can be seen that $\rho_1 = \frac{2}{5}$ and $\rho_i = \frac{1}{4i-2}$ for $i \geq 2$ and that $\min\{c,\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{4i-2} \geq 1\} = 11$. For the upper bound, a 16-packing coloring of period 320 is given in Appendix B2.

For $D(1,5)$, the computer tells us that there exists no 9-packing coloring of $D_{45}(1,5)$ and a 12-packing coloring of period 1028 is given in [**?**].

Similarly, for $D(1,6)$, there exists no 10-packing coloring of $D_{45}(1,6)$ and a 23-packing coloring of period 1917 is given in [**?**]. For $D(1,7)$, there exists no 9-packing coloring of $D_{45}(1,7)$ and a 15-packing coloring of period 640 is given in [**?**]. For $D(1,8)$, there exists no 10-packing coloring of $D_{45}(1,8)$ and a 25-packing coloring of period 5184 is given in [**?**]. For $D(1,9)$, there exists no 9-packing coloring of $D_{45}(1,7)$ and a 18-packing coloring of period 576 is given in [**?**].

□

It is interesting to notice that sometimes adding just one more color allows to shorten considerably the period of the packing coloring, as can be seen with $D(1,5)$ with the following periodic 13-packing coloring of period 80 (compared with the 12-packing coloring of period 1028):
$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,10,1,4,1,2,1,3,1,5,1,11,1,2,1,3,1,8,1,9,$
$1,2,1,3,1,4,1,5,1,2,1,3,1,6,1,7,1,2,1,3,1,12,1,4,1,2,1,3,1,5,1,13,1,2,1,3,1,9,1,8.$

We now turn our attention on other distance graphs with two chords, i.e. graphs of type $D(a,b)$, with $2 \leq a \leq b$. The smallest example is $D(2,3)$ which is a subgraph of $D(1,2,3) = P_\infty^3$, thus $\chi_\rho(D(2,3)) \leq \chi_\rho(P_\infty^3) \leq 23$. In fact, we show that the upper bound is much less than 22:

Proposition 11.

$$11 \leq \chi_\rho(D(2,3)) \leq 13;$$

$$\chi_\rho(D(2,4)) = 8;$$

$$14 \leq \chi_\rho(D(2,5)) \leq 23;$$

$$\chi_\rho(D(2,6)) = 9.$$

Proof. The graph $D(2,4)$ ($D(2,6)$, respectively) is not connected and consists in two copies of $D(1,2)$ ($D(1,3)$, respectively). Thus $\chi_\rho(D(2,4)) = \chi_\rho(D(1,2)) = 8$ and $\chi_\rho(D(2,6)) = \chi_\rho(D(1,3)) = 9$.

The lower bound $11 \leq \chi_\rho(D(2,3))$ is obtained by calculating the maximum density ρ_i of a color i: it can be seen that $\rho_1 = \frac{2}{5}$ and $\rho_i = \frac{1}{3i+1}$ for $i \geq 2$ and that $\min\{c,\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{3i+1} \geq 1\} = 11$.

For the lower bound $14 \leq \chi_\rho(D(2,5))$, it can be seen that $\rho_1 = \frac{3}{7}$ and $\rho_i = \frac{1}{5i-4}$ for $i \geq 2$ and that $\min\{c,\ \frac{2}{5} + \sum_{i=2}^{c} \frac{1}{5i-4} \geq 1\} = 14$.

The upper bound $\chi_\rho(D(2,3)) \leq 13$ comes from the 13-packing coloring of period 240 given in Appendix B3 and the bound $\chi_\rho(D(2,5)) \leq 23$ comes from the 23-packing coloring of period 336 given in Appendix B4. □

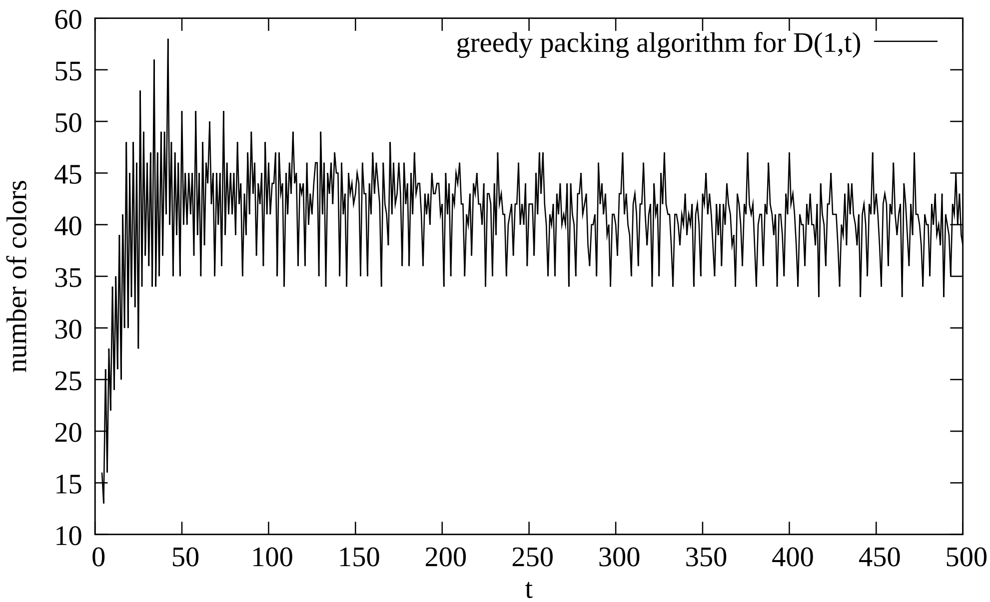

Figure 5: Number of colors for a packing coloring of $D(1, t)$ using a greedy algorithm.

5 Concluding remarks

We have shown that the packing chromatic number of any infinite distance graph with finite D is finite and is at most 40 (81, respectively) for $D = \{1, t\}$ with t being an odd (even, respectively) integer greater than or equal to 447.

Among the many directions in this area, one can try to find better bounds and/or more simple methods for $D(1, t)$. In fact, running a simple greedy packing coloring algorithm that consists in coloring vertices of a distance graph one-by-one from the left to the right with the smallest color with respect to the constraint, suggests that the upper bounds found in Section 3 can be strengthened. Figure 5 shows the number of colors used by the greedy algorithm for a packing coloring of $D_n(1, t)$ (with $n = 1000000$) as a function of t for the first 500 values of t.

One can see on the figure that for large t, the algorithm finds a packing coloring, using between 30 and 50 colors. Moreover, more colors are needed in general when t is even compared to when t is odd. But surprisingly, even if we look only at even (or odd) values of t, the function is not monotonic. We wonder if the same goes for χ_ρ. An interesting future work would be to study more in details the behavior of this greedy algorithm.

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Appendix A: Modification of the color patterns depending on the residues of s modulo 48

For $s \equiv 0$ or $4 \bmod 48$, the color patterns are given in proof of Proposition 6, so only the modifications to obtain the color patterns for other residues of s are presented here.

$s \bmod 48$	8	12	16
P_2	$+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	$+4,5,8,4,5,9$	idem $+\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_2'	$+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	$+4,5,10,4,5,11$	idem $+\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$
P_3	$+6,7,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}32,35,38,41\\33,36,39,42\\34,37,40,43\end{smallmatrix}\right.$	$+6,7,12,13,6,7,14,15$
P_3'	$+6,7,\left\{\begin{smallmatrix}51,53\\52,54\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}44,46,48,50\\45,47,49,51\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}44,46\\45,47\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}48,50\\49,51\end{smallmatrix}\right.$
P_3''	$+6,7,\left\{\begin{smallmatrix}55,56\\57,58\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}52,54,56,58\\53,55,57,59\end{smallmatrix}\right.$	$+6,7,\left\{\begin{smallmatrix}52,54\\53,55\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}56,58\\57,59\end{smallmatrix}\right.$

$s \bmod 48$	20	24
P_2	idem $s \equiv 12$ $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	$+(4,5,8,4,5,9)^2$
P_2'	idem $s \equiv 12$ $+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	$+(4,5,10,4,5,11)^2$
P_3	idem $+\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$	idem $s \equiv 16$ $+6,7,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_3'	$+6,7,\left\{\begin{smallmatrix}51,53\\52,54\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}55,57,59,61\\56,58,60,62\end{smallmatrix}\right.$	0
P_3''	$+6,7,\left\{\begin{smallmatrix}63,66\\64,67\\65,68\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}69,72,75,78\\70,73,76,79\\71,74,77,80\end{smallmatrix}\right.$	$+6,7,22,23,6,7,\left\{\begin{smallmatrix}38,40\\39,41\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}42,44\\43,45\end{smallmatrix}\right.$

$s \bmod 48$	28	32	36
P_2	idem $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	idem $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	$+(4,5,8,4,5,9)^3$
P_2'	idem $+\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	idem $s \equiv 24$ $+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	$+(4,5,10,4,5,11)^3$
P_3	$+6,7,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47,51,55\\48,52,56\\49,53,57\\50,54,58\end{smallmatrix}\right.$	$+(6,7,12,13,6,7,14,15)^2$	idem $+\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_3'	idem $+\left\{\begin{smallmatrix}59,61\\60,62\end{smallmatrix}\right.$	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}44,46\\45,47\end{smallmatrix}\right.$	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}38,40,42,44\\39,41,43,45\end{smallmatrix}\right.$
P_3''	idem $+\left\{\begin{smallmatrix}63,66\\64,67\\65,68\end{smallmatrix}\right.$	0	idem $+\left\{\begin{smallmatrix}46,48\\47,49\end{smallmatrix}\right.$

$s \bmod 48$	40	44
P_2	idem $+\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$	idem $s \equiv 36$ $+4,5,\left\{\begin{smallmatrix}32,35\\33,36\\34,37\end{smallmatrix}\right.$
P_2'	idem $s \equiv 32$ $+\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$	idem $s \equiv 36$ $+4,5,\left\{\begin{smallmatrix}38,41\\39,42\\40,43\end{smallmatrix}\right.$
P_3	idem $+6,7,\left\{\begin{smallmatrix}44\\45\\46\end{smallmatrix}\right.,\left\{\begin{smallmatrix}47\\48\\49\\50\end{smallmatrix}\right.$	idem $+\left\{\begin{smallmatrix}51,55\\52,56\\53,57\\54,58\end{smallmatrix}\right.$
P_3'	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}51,53\\52,54\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}55,57\\56,58\end{smallmatrix}\right.$	idem $s \equiv 24$ $+6,7,\left\{\begin{smallmatrix}59,61\\60,62\end{smallmatrix}\right.,6,7,\left\{\begin{smallmatrix}63,66,69,72\\64,67,70,73\\65,68,71,74\end{smallmatrix}\right.$
P_3''	idem $s \equiv 32$ $+6,7,\left\{\begin{smallmatrix}59,61\\60,62\end{smallmatrix}\right.$	idem $s \equiv 32$ $+6,7,\left\{\begin{smallmatrix}75,78,81,84\\76,79,82,85\\77,80,83,86\end{smallmatrix}\right.$

Appendix B1: A periodic 23-packing coloring of P_∞^3 of period 768

23, 1, 4, 5, 3, 1, 2, 6, 7, 1, 9, 10, 12, 1, 2, 3, 4, 1, 8, 5, 13, 1, 2, 14, 16, 1, 3, 6, 11, 1, 2, 4, 7, 1, 15, 5, 3, 1, 2, 9, 18, 1, 10, 8, 4, 1, 2, 3, 6, 1, 12, 5, 17, 1, 2, 7, 19, 1, 3, 4, 13, 1, 2, 11, 20, 1, 14, 5, 3, 1, 2, 6, 4, 1, 8, 9, 10, 1, 2, 3, 7, 1, 15, 5, 16, 1, 2, 4, 12, 1, 3, 6, 21, 1, 2, 18, 22, 1, 11, 5, 3, 1, 2, 4, 7, 1, 8, 9, 10, 1, 2, 3, 6, 1, 13, 5, 4, 1, 2, 14, 17, 1, 3, 19, 23, 1, 2, 7, 12, 1, 4, 5, 3, 1, 2, 6, 8, 1, 9, 10, 11, 1, 2, 3, 4, 1, 15, 5, 16, 1, 2, 7, 18, 1, 3, 6, 13, 1, 2, 4, 20, 1, 8, 5, 3, 1, 2, 9, 12, 1, 10, 14, 4, 1, 2, 3, 6, 1, 7, 5, 11, 1, 2, 17, 19, 1, 3, 4, 8, 1, 2, 21, 15, 1, 22, 5, 3, 1, 2, 6, 4, 1, 7, 9, 10, 1, 2, 3, 12, 1, 13, 5, 16, 1, 2, 4, 8, 1, 3, 6, 11, 1, 2, 14, 7, 1, 18, 5, 3, 1, 2, 4, 9, 1, 20, 10, 17, 1, 2, 3, 6, 1, 8, 5, 4, 1, 2, 7, 12, 1, 3, 13, 15, 1, 2, 11, 19, 1, 4, 5, 3, 1, 2, 6, 9, 1, 10, 8, 14, 1, 2, 3, 4, 1, 7, 5, 16, 1, 2, 21, 22, 1, 3, 6, 18, 1, 2, 4, 12, 1, 11, 5, 3, 1, 2, 8, 7, 1, 9, 10, 4, 1, 2, 3, 6, 1, 13, 5, 15, 1, 2, 14, 17, 1, 3, 4, 19, 1, 2, 7, 8, 1, 20, 5, 3, 1, 2, 6, 4, 1, 9, 10, 11, 1, 2, 3, 12, 1, 16, 5, 18, 1, 2, 4, 7, 1, 3, 6, 8, 1, 2, 13, 21, 1, 14, 5, 3, 1, 2, 4, 9, 1, 10, 15, 17, 1, 2, 3, 6, 1, 7, 5, 4, 1, 2, 8, 11, 1, 3, 12, 19, 1, 2, 20, 22, 1, 4, 5, 3, 1, 2, 6, 7, 1, 9, 10, 13, 1, 2, 3, 4, 1, 8, 5, 14, 1, 2, 16, 18, 1, 3, 6, 11, 1, 2, 4, 7, 1, 12, 5, 3, 1, 2, 9, 15, 1, 10, 8, 4, 1, 2, 3, 6, 1, 17, 5, 13, 1, 2, 7, 19, 1, 3, 4, 20, 1, 2, 11, 14, 1, 21, 5, 3, 1, 2, 6, 4, 1, 8, 9, 10, 1, 2, 3, 7, 1, 12, 5, 16, 1, 2, 4, 15, 1, 3, 6, 13, 1, 2, 18, 22, 1, 11, 5, 3, 1, 2, 4, 7, 1, 8, 9, 10, 1, 2, 3, 6, 1, 14, 5, 4, 1, 2, 12, 17, 1, 3, 19, 20, 1, 2, 7, 23, 1, 4, 5, 3, 1, 2, 6, 8, 1, 9, 10, 11, 1, 2, 3, 4, 1, 13, 5, 15, 1, 2, 7, 16, 1, 3, 6, 12, 1, 2, 4, 14, 1, 8, 5, 3, 1, 2, 9, 18, 1, 10, 21, 4, 1, 2, 3, 6, 1, 7, 5, 11, 1, 2, 17, 19, 1, 3, 4, 8, 1, 2, 13, 20, 1, 12, 5, 3, 1, 2, 6, 4, 1, 7, 9, 10, 1, 2, 3, 14, 1, 15, 5, 16, 1, 2, 4, 8, 1, 3, 6, 11, 1, 2, 18, 7, 1, 22, 5, 3, 1, 2, 4, 9, 1, 12, 10, 13, 1, 2, 3, 6, 1, 8, 5, 4, 1, 2, 7, 17, 1, 3, 14, 19, 1, 2, 11, 15, 1, 4, 5, 3, 1, 2, 6, 9, 1, 10, 8, 16, 1, 2, 3, 4, 1, 7, 5, 12, 1, 2, 13, 18, 1, 3, 6, 20, 1, 2, 4, 21, 1, 11, 5, 3, 1, 2, 8, 7, 1, 9, 10, 4, 1, 2, 3, 6, 1, 14, 5, 15, 1, 2, 17, 19, 1, 3, 4, 12, 1, 2, 7, 8, 1, 13, 5, 3, 1, 2, 6, 4, 1, 9, 10, 11, 1, 2, 3, 16, 1, 18, 5, 22, 1, 2, 4, 7, 1, 3, 6, 8, 1, 2, 14, 20, 1, 12, 5, 3, 1, 2, 4, 9, 1, 10, 13, 15, 1, 2, 3, 6, 1, 7, 5, 4, 1, 2, 8, 11, 1, 3, 17, 19, 1, 2, 21

Appendix B2: A periodic 16-packing coloring of $D(1,4)$ of period 320

1, 2, 1, 3, 4, 1, 5, 1, 2, 7, 1, 6, 1, 3, 2, 1, 8, 1, 4, 10, 1, 2, 1, 3, 5, 1, 9, 1, 2, 12, 1, 13, 1, 3, 2, 1, 4, 1, 6, 7, 1, 2, 1, 3, 11, 1, 5, 1, 2, 8, 1, 4, 1, 3, 2, 1, 14, 1, 10, 15, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 7, 1, 3, 2, 1, 9, 1, 12, 8, 1, 2, 1, 3, 4, 1, 5, 1, 2, 11, 1, 6, 1, 3, 2, 1, 10, 1, 4, 13, 1, 2, 1, 3, 5, 1, 7, 1, 2, 8, 1, 9, 1, 3, 2, 1, 4, 1, 6, 14, 1, 2, 1, 3, 12, 1, 5, 1, 2, 15, 1, 4, 1, 3, 2, 1, 7, 1, 10, 8, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 9, 1, 3, 2, 1, 11, 1, 13, 16, 1, 2, 1, 3, 4, 1, 5, 1, 2, 7, 1, 6, 1, 3, 2, 1, 8, 1, 4, 10, 1, 2, 1, 3, 5, 1, 9, 1, 2, 12, 1, 14, 1, 3, 2, 1, 4, 1, 6, 7, 1, 2, 1, 3, 11, 1, 5, 1, 2, 8, 1, 4, 1, 3, 2, 1, 13, 1, 10, 15, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 7, 1, 3, 2, 1, 9, 1, 12, 8, 1, 2, 1, 3, 4, 1, 5, 1, 2, 11, 1, 6, 1, 3, 2, 1, 10, 1, 4, 14, 1, 2, 1, 3, 5, 1, 7, 1, 2, 8, 1, 9, 1, 3, 2, 1, 4, 1, 6, 13, 1, 2, 1, 3, 12, 1, 5, 1, 2, 15, 1, 4, 1, 3, 2, 1, 7, 1, 10, 8, 1, 2, 1, 3, 5, 1, 4, 1, 2, 6, 1, 9, 1, 3, 2, 1, 11, 1, 14, 16

Appendix B3 : A periodic 13-packing coloring of $D(2,3)$ of period 240

1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 13, 1, 1, 2, 4, 11, 1, 1, 7, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 8, 10, 2, 1, 1, 12, 3, 5, 1, 1, 2, 4, 6, 1, 1, 7, 3, 2, 1, 1, 9, 11, 13, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 7, 1, 1, 2, 4, 10, 1, 1, 12, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 7, 8, 2, 1, 1, 11, 3, 5, 1, 1, 2, 4, 6, 1, 1, 10, 3, 2, 1, 1, 9, 13, 7, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 12, 1, 1, 2, 4, 11, 1, 1, 7, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 8, 10, 2, 1, 1, 13, 3, 5, 1, 1, 2, 4, 6, 1, 1, 7, 3, 2, 1, 1, 9, 11, 12, 1, 1, 2, 3, 4, 1, 1, 5, 6, 2, 1, 1, 8, 3, 7, 1, 1, 2, 4, 10, 1, 1, 13, 3, 2, 1, 1, 5, 6, 9, 1, 1, 2, 3, 4, 1, 1, 7, 8, 2, 1, 1, 11, 3, 5, 1, 1, 2, 4, 6, 1, 1, 10, 3, 2, 1, 1, 9, 12, 7

Appendix B4 : A periodic 23-packing coloring of $D(2,5)$ of period 336

1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 7, 8, 1, 1, 2, 2, 1, 3, 10, 1, 1, 11, 4, 1, 15, 12, 1, 1, 2, 2, 1, 3, 16, 1, 1, 5, 6, 1, 4, 9, 1, 1, 2, 2, 1, 3, 7, 1, 1, 8, 14, 1, 17, 13, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 19, 1, 1, 2, 2, 1, 3, 11, 1, 1, 7, 4, 1, 9, 12, 1, 1, 2, 2, 1, 3, 8, 1, 1, 5, 6, 1, 4, 15, 1, 1, 2, 2, 1, 3, 18, 1, 1, 20, 21, 1, 7, 22, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 9, 1, 1, 2, 2, 1, 3, 8, 1, 1, 11, 4, 1, 13, 12, 1, 1, 2, 2, 1, 3, 7, 1, 1, 5, 6, 1, 4, 14, 1, 1, 2, 2, 1, 3, 16, 1, 1, 17, 23, 1, 9, 19, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 7, 8, 1, 1, 2, 2, 1, 3, 10, 1

, 1, 11, 4, 1, 15, 12, 1, 1, 2, 2, 1, 3, 13, 1, 1, 5, 6, 1, 4, 9, 1, 1, 2, 2, 1, 3, 7, 1, 1, 8, 18, 1, 14, 20, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 21, 1, 1, 2, 2, 1, 3, 11, 1, 1, 7, 4, 1, 9, 12, 1, 1, 2, 2, 1, 3, 8, 1, 1, 5, 6, 1, 4, 13, 1, 1, 2, 2, 1, 3, 15, 1, 1, 16, 17, 1, 7, 19, 1, 1, 2, 2, 1, 3, 4, 1, 1, 5, 6, 1, 10, 9, 1, 1, 2, 2, 1, 3, 8, 1, 1, 11, 4, 1, 14, 12, 1, 1, 2, 2, 1, 3, 7, 1, 1, 5, 6, 1, 4, 18, 1, 1, 2, 2, 1, 3, 13, 1, 1, 20, 22, 1, 9, 23